Focusing of spherical Gaussian beams

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Simple procedures and formulas for tracing the characteristics of a spherical Gaussian beam through a train of lenses or mirrors are described which are analogous to those used in geometrical optics to trace repeated images through an optical train.

I. Introduction

In laboratory experiments with lasers, it is frequently necessary to follow the properties of a Gaussian beam after passage through a number of lenses. Often one is required to design a lens system to create a beam waist of specified diameter at a specified location, for example, in laser anemometry.

Because on the laboratory scale one is often working with a lens in the near field of the incident beam, the behavior of the beam can be significantly different from that which would be anticipated on the basis of geometrical optics.

An extreme example of the difference in behavior between Gaussian beams and conventional uniform spherical waves from point sources occurs when the waist of the incident beam is at the front focal plane of a positive lens, in which case the emerging beam has a waist at the back focal plane. This is inexplicable on the basis of geometrical optics, which predicts that a point object at the front focus yields a collimated beam in the image space, i.e., the image is at infinity. Another example, concerned with the focal shift, i.e., the difference between the position of the image waist for a Gaussian beam from its geometrical optics position, is discussed in a recent paper by Carter,[1] who works in terms of the Fresnel number at the lens.

Siegman[2] discusses the properties of Gaussian beams and treats the problem of following the transformation of such a beam under the action of a train of lenses in a general manner using matrix algebra.

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In this laboratory, a simple procedure for tracing the characteristics of a Gaussian beam through a train of lenses (or mirrors) was devised[3] some years ago and has proved useful. It generalizes the lens (or mirror) formulas of geometrical optics by introducing an additional parameter, the Rayleigh range of the beam which can be simply recalculated for each step. Its main virtue lies in the fact that the procedure is analogous to that traditionally used for following repeated images through an optical train on the basis of geometrical optics but with modified formulas. The Rayleigh range, a property of the beam incident on each lens, appears to be a more convenient parameter than the Fresnel number used by Carter.[1]

This paper describes the procedure and formulas adopted and brings out the connection with geometrical optics. The discussion is given in terms of lenses, but with the appropriate sign convention, the results apply equally well for mirrors.

II. Focusing Uniform Spherical Waves

The standard treatment[4] of thin lenses (or mirrors) by geometrical optics neglects diffraction and deals with point objects and images and uniform spherical waves whose radii of curvature equal the distances from the point object or image.

The change in wave-front radius of curvature introduced by a lens of focal length f is just $1/f$ (with a suitable sign convention and assuming identical media on either side). With the sign conventions of Ref. 4, the standard thin lens or mirror formula is

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}, \tag{1}$$

where s, s' are the object and image distances, respectively.

This can be written in dimensionless form

$$\frac{1}{(s/f)} + \frac{1}{(s'/f)} = 1 \tag{1a}$$

and is illustrated in the Cartesian plot of Fig. 1.

In this form, the lens equation plots as a rectangular hyperbola with asymptotes $(s/f) = 1, (s'/f) = 1$. Ray diagrams representing the situations for branch (1) and the segments (2) and (3) of the other branch of the hyperbola are shown for the cases of both positive ($f > 0$) and negative ($f < 0$) lenses.

The magnification taken as positive,

$$m = \left|\frac{s'}{s}\right| = \frac{1}{|1 - (s/f)|}, \quad (2)$$

is shown as a broken line in Fig. 1. Thus, Fig. 1 effectively summarizes all the properties and uses of a simple lens in image forming.

III. Properties of a Spherical Gaussian Beam

The TEM_{00} mode output of a laser is a spherical Gaussian beam which has a waist, either real or, more commonly, virtual (by projection backward inside the laser), where the wave front is planar, and the beam diameter is a minimum.

With respect to cylindrical coordinates with origin at the waist (Fig. 2) and with Siegman's[2] notation, the intensity distribution normalized to unit total beam power is

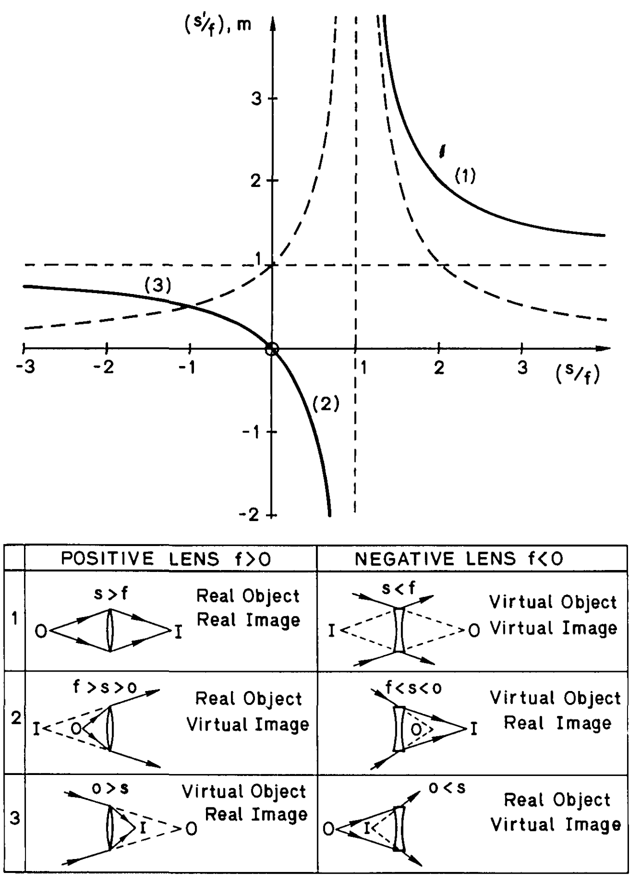

Fig. 1. Cartesian plot of the lens formula of geometrical optics. Solid lines show normalized image distance vs normalized object distance. Broken lines show the magnification. Below are sketched ray diagrams illustrating the situations for points on branch (1) and on the segments (2) and (3) of the second branch.

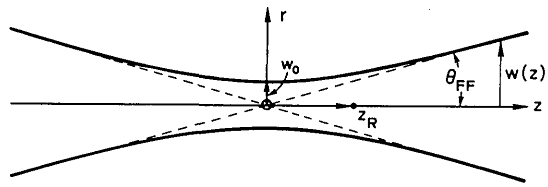

Fig. 2. Geometry of a spherical Gaussian beam.

$$I(r,z) = (2/\pi w^2) \exp - 2(r/w)^2, \quad (3)$$

where the beam radius w (to $1/e^2$ of the intensity on axis) is

$$w(z) = w_0[1 + (z/z_R)^2]^{1/2}. \quad (4)$$

Here w_0 is the beam radius at the waist ($z = 0$), and

$$z_R \equiv (\pi w_0^2/\lambda) \quad (5)$$

is the Rayleigh range characterizing the Lorentzian profile of intensity along the axis.

The near and far fields are defined, respectively, by $z < z_R$ and $z > z_R$.

The radius of curvature of the wave fronts is

$$R(z) = z[1 + (z_R/z)^2]. \quad (6)$$

In the near field, for $z \ll z_R$, $R \sim z_R^2/z \to \infty$ for $z \to 0$. In the far field for $z \gg z_R$, $R \sim z$, and the waves approximate those from a point source centered at the waist. It may be noted that the radius of curvature has a minimum $R_{\min} = 2z_R$ at $z = z_R$.

In the far field the beam radius is

$$w_{FF} \to w_0(z/z_R) = \lambda z/\pi w_0. \quad (7)$$

Hence, in the far field the half-angle of divergence is

$$\theta_{FF} = w_{FF}/z = \lambda/\pi w_0. \quad (8)$$

It may be noted that w_0 and λ determine all the beam properties. More generally, two quantities, e.g., $w(z_1)$, $w(z_2)$; $R(z_1)$, $R(z_2)$; $w(z_1)$, $R(z_1)$ or $w(z_1)$, θ_{FF}, suffice to determine the beam properties and waist position. From the expressions quoted, the location and radius of the beam waist can be calculated from any two such quantities.

IV. Focusing Spherical Gaussian Beams

For comparison with the geometrical optics case in calculating the focusing effect of a thin lens, we regard the waist of the input beam as the object and the waist of the output beam as the image. Either may be real or virtual. Also, the ratio of the output/input waist diameters is the magnification. The situation for a positive lens with real object and image beam waists is shown in Fig. 3.

If the input beam waist radius w_0 and the object distance s are specified, the Rayleigh range and the beam's radius $w(s)$ and radius of curvature $R(s)$ at the lens can be calculated from Eqs. (5), (4), and (6), respectively.

For a thin lens, the beam radius is unchanged through the lens, while the radius of curvature is changed by an

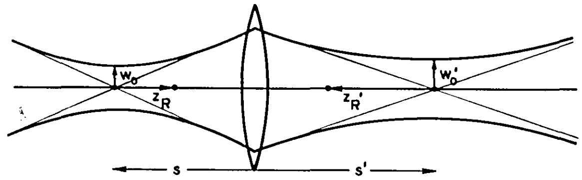

Fig. 3. Geometry of the imaging of a Gaussian beam by a lens shown for the case of a positive lens and real object and image waists.

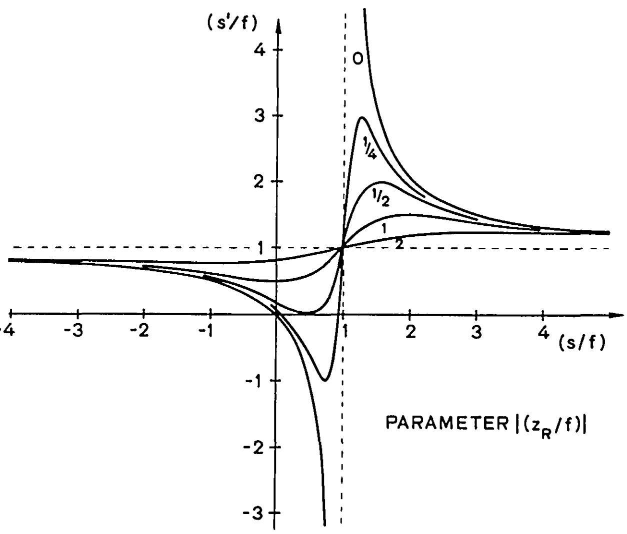

Fig. 4. Cartesian plot of the lens formula for Gaussian beams showing normalized image distance vs normalized object distance, with normalized Rayleigh range of the input beam as the parameter.

amount $(1/f)$ as in the geometrical optics case. This is sufficient to determine the characteristics of the output beam.

Applying the formulas (4)–(6) then yields the following lens formula for a Gaussian beam

$$\frac{1}{s + z_R^2/(s - f)} + \frac{1}{s'} = \frac{1}{f}. \tag{9}$$

In normalized form this becomes

$$\frac{1}{(s/f) + (z_R/f)^2/(s/f - 1)} + \frac{1}{(s'/f)} = 1, \tag{9a}$$

which may be compared with Eq. (1a). Regarding (s'/f) as a function of (s/f), this can be written

$$(s'/f) = 1 + \frac{[(s/f) - 1]}{[(s/f) - 1]^2 + (z_R/f)^2}, \tag{9b}$$

whereas the equivalent form of the usual lens formula is

$$(s'/f) = 1 + \frac{1}{[(s/f) - 1]}. \tag{1b}$$

It is clear that the appearance of the term $(z_R/f)^2$ in the denominator of Eq. (9b) removes the pole at $(s/f) = 1$ appearing in the usual lens formula (1b).

The lens formula for Gaussian beams is plotted in normalized form in Fig. 4, with (z_R/f) as parameter. The limit $(z_R/f) \to 0$ corresponds to the geometrical optics case.

It is particularly noteworthy that, for nonzero values of (z_R/f), all the curves have a single continuous branch passing through the point $(s/f) = (s'/f) = 1$, where there is a point of inflection. The asymptote at $(s/f) = 1$ has been removed, which means that the image distance cannot become infinite as in geometrical optics, when the object lies at a focal point.

The common point $(s/f) = (s'/f) = 1$ represents the fact noted earlier that, if the incident beam waist lies at the front focus, the emerging beam has a waist at the back focus. Moreover, this is independent of the ratio (z_R/f). For a negative lens, the common point corresponds to the incident beam having a virtual waist at the back focus and the emerging beam having a virtual waist at the front focus. Neither of these situations has a counterpart in geometrical optics.

The maxima and minima of the curves are also of interest. By differentiation we find

$$(s'/f)_{\max} = 1 + \tfrac{1}{2}(z_R/f) \text{ at } (s/f) = 1 + (z_R/f); \tag{10a}$$

$$(s'/f)_{\min} = 1 - \tfrac{1}{2}(z_R/f) \text{ at } (s/f) = 1 - (z_R/f). \tag{10b}$$

For the common case of a positive lens and real object and image, Eq. (10a) says that the maximum image distance is $s'_{\max} = f + f^2/2z_R$, and this occurs for an object distance $s = f + z_R$.

Unlike the case of geometrical objects, where for real object and image there is a minimum object to image distance $s + s' = 4f$ (when $s = s' = 2f$, and the magnification is unity), there is no corresponding minimum separation between the waists of the input and output beams for Gaussian beams.

The magnification is given by

$$m = \frac{w_0'}{w_0} = \frac{1}{\{[1 - (s/f)]^2 + (z_R/f)^2\}^{1/2}}, \tag{11}$$

which reduces to the geometrical optics formula (2) when $z_R^2 \ll (s - f)^2$. The presence of the term $(z_R/f)^2$ removes the pole in the magnification when $s = f$ in Eq. (2). For the latter condition, when the object waist is at the front focus, the magnification has a maximum of (f/z_R); unity magnification only occurs when $f = z_R$ for this case.

The magnification as a function of (s/f) is graphed in Fig. 5, with (z_R/f) as parameter. The limit $(z_R/f) \to 0$ corresponds to the geometrical optics case shown as a broken line in Fig. 1.

The Rayleigh range of the output beam is given by

$$z_R' = m^2 z_R. \tag{12}$$

Thus, from Eqs. (9b), (11), and (12), all the properties of the output beam, i.e., the normalized waist (image) distance s'/f, the magnification m, and the Rayleigh range z_R', are calculable from the normalized object distance (s/f) and the properties of the input beam specified by the single parameter z_R. For a given focal length and object distance, Figs. 4 and 5 allow one to

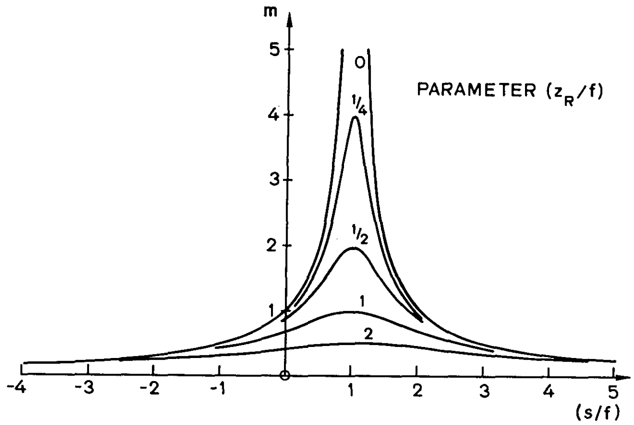

Fig. 5. Graph of magnification for a Gaussian beam vs normalized object distance, with normalized Rayleigh range of the input beam as the parameter.

quickly assess whether one is working near the geometrical optics limit or if there are substantial corrections due to working in the near field.

V. Discussion

Because Eq. (9) has been written in terms of the Rayleigh range of the input beam, it is not symmetrical with respect to input and output quantities, as is the geometrical optics formula. However, the reciprocal relation in terms of the Rayleigh range of the output beam can be written

$$\frac{1}{s} + \frac{1}{s' + z_R'^2/(s' - f)} = \frac{1}{f} \,. \qquad (13)$$

This is useful for tracing backward to find the properties of the input beam when those of the output beam are known.

It may also be noted that Eq. (9) is used with the usual sign convention for s, s', and f and with the conventional distinctions between real and virtual objects and images.

Commonly, in tracing the beam through an optical system or designing a lens system to create a beam with required properties, one has to start with the properties of the beam emitted by the laser. Thus, one needs to know the effective waist position and Rayleigh range of the laser beam.

If the output mirror is planar, one knows that the waist is at the output mirror. Usually the manufacturer specifies the output beam diameter and far-field divergence angle, but it has been the author's experience that these are not always as stated or known accurately enough to determine the Rayleigh range with sufficient precision. Where possible, it is better to calculate the theoretical waist position and diameter from the radii of curvature and separation of the mirrors using the cavity mode equations.[2] When the output mirror is not planar, allowance should be made for the lens effect of the output mirror since it usually has a planar second surface. This can be done using the method discussed above to find the position of the virtual waist and the Rayleigh range of the output beam.

Finally, it should be noted that, throughout this paper, it has been assumed that the lens diameter is sufficient to not significantly aperture the Gaussian beam. If significant aperturing effect occurs, the image position will not be as given above, and the waist will not have a truly Gaussian profile. In the limit of a strong aperturing effect, the beam at the lens will approach the uniform spherical wave assumed in geometrical optics, and the image waist will have the usual Airy diffraction pattern and will occur at the geometrical optics image position.

References

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4. F. A. Jenkins and H. E. White, *Fundamentals of Optics* (McGraw-Hill, New York, 1957), Chap. 4.