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NEW TRENDS IN FLUID AND SOLID MECHANICAL MODELS

Fractional models of anomalous relaxation based on the Kilbas and Saigo function

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Abstract We revisit the Kilbas and Saigo functions of the Mittag-Leffler type of a real variable t, with two independent real order-parameters. These functions, subjected to the requirement to be completely monotone for $t > 0$, can provide suitable models for the responses and for the corresponding spectral distributions in anomalous (non–Debye) relaxation processes, found e.g. in dielectrics. Our analysis includes as particular cases the classical models referred to as Cole–Cole (the one-parameter Mittag-Leffler function) and to as Kohlrausch (the stretched exponential function). After some remarks on the Kilbas and Saigo functions, we discuss a class of fractional differential equations of order $\alpha \in (0, 1]$ with a characteristic coefficient varying in time according to a power law of exponent β, whose solutions will be presented in terms of these functions. We show 2D plots of the solutions and, for a few of them, the corresponding spectral distributions, keeping fixed one of the two order-parameters. The numerical results confirm the complete monotonicity of the solutions via the non-negativity of the spectral distributions, provided that the parameters satisfy the additional condition $0 < \alpha + \beta \leq 1$, assumed by us.

Keywords Anomalous relaxation · Completely monotone functions · Fractional derivative · Spectral distributions · Mittag-Leffler functions

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1 Introduction

In a recent paper Capelas de Oliveira et al. [3] revisited the Mittag-Leffler functions of a real variable t, with one, two and three order-parameters $\{\alpha, \beta, \gamma\}$, as far as their Laplace transform pairs and complete monotonicity properties are concerned. These functions, subjected to the requirement to be completely monotone for $t > 0$, are shown to be suitable models for the physical realizability of non–Debye (or anomalous) relaxation phenomena in dielectrics including as particular cases the classical models referred to as Cole and Cole [4, 5], Davidson and Cole [6] and Havriliak and Negami [16, 17]. In the literature a number of laws have been proposed to describe the non–Debye relaxation phenomena in dielectrics, of which the most relevant are the ones referred above, along with the so-called Kohlrausch law or Kohlrausch and Williams-Watts (KWW) [35, 56] law (based on the stretched exponential function). For

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more details see e.g. the classical books by Jonscher [21,22] and the recent book by Uchaikin and Sibatov [53]. Several authors have investigated these laws and possible generalizations from different points of view, including Hilfer [18, 19], Jurlewicz and Weron [23], Anderssen et al. [2], and more recently Hanyga and Seredyńska [14], Jurlewicz et al. [24] and Stanislavski et al. [52], Weron et al. [54], just to cite a few.

On the other hand, in our opinion other functions that can be related to anomalous relaxation processes are the so-called Kilbas and Saigo functions of Mittag-Leffler type of a real variable t, with two independent real order-parameters [29]. In fact these functions, subjected to the requirement to be completely monotone for $t > 0$, will be shown to provide suitable models for the responses in non–Debye relaxation processes found e.g. in dielectrics, including as particular cases the classical model of Cole–Cole (the one-parameter Mittag-Leffler function) and the Kohlrausch model (the stretched exponential function).

This paper is organized as follows.

In Sect. 2, an overview of the classical Mittag-Leffler functions and Kilbas and Saigo function is presented.

Section 3 is devoted to present the fractional differential equations of relaxation type with non constant coefficients whose solutions are expressed in terms of Kilbas and Saigo function depending on two-order parameters. As noteworthy particular cases we recover the classical Mittag-Leffler function and the stretched exponential function.

In Sect. 4, we show the complete monotonicity of the Kilbas and Saigo function by presenting the corresponding non-negative spectral distribution in the form of a power series. This will be achieved by using standard methods for Laplace and Stieltjes transforms. As in Sect. 3, we also recover the particular cases concerning the spectral distribution associated with the clasical Mittag-Leffler function and with the stretched exponential function.

Section 5 is devoted to the numerical results. For some selected values of the two order parameters we provide 2D plots of the responses functions and of the spectral distributions, in order to better visualize the positivity and the variability of the considered functions.

Finally, Sect. 6 is devoted to the concluding remarks.

In the appendices, that close the paper, we provide the details for deriving the noteworthy formulas presented in Sect. 4.

2 An overview on the Mittag-Leffler functions

In 1903 Mittag-Leffler published a paper in which a generalization of the exponential function containing a single parameter was introduced [41] and named after him Mittag-Leffler function. Two years later he published a complete study of this function [42].

A first generalization of this function was presented by Wiman [57] and studied by Agarwal [1] and Humbert and Agarwal [20] with the addition of another parameter. This generalization is referred to as two-parameter Mittag-Leffler function. The one-parameter and two-parameter Mittag-Leffler functions appear also in the classical Bateman hand-book [8].

Later on, some ways emerge in the literature in order to generalize the previous Mittag-Leffler functions with additional parameters. The most famous and simple is the so-called three-parameter Mittag-Leffler function as proposed in 1971 by Prabhakar [46] also named after him.

We point out that these three functions have nice properties of complete monotonicity for negative real argument according to some known relations among the parameters as formerly showed by Pollard [45] for the standard Mittag-Leffler function, by Schneider [49] and Miller and Samko [39] for the two parameter function and, more recently, by Capelas et al. [3] for the Prabhakar function. Se also Appendix E in the recent book by Mainardi [38] and references therein.

In this paper we restrict our attention to three-parameter functions of the Mittag-Leffler type different from the Prabhakar functions and to their application in fractional differential equations related to phenomena of non standard relaxation. We note that these generalized Mittag-Leffler functions were proposed for the first time in 1995 by Kilbas and Saigo in relation to solutions of non-linear integral equations of Abel-Volterra type [26–28] and will be referred to as Kilbas and Saigo functions in the following. In [29] a relation of this generalized Mittag-Leffler function to fractional calculus was discussed and in [30] a class of linear differential equations of fractional order was solved in a closed form, see also the paper by Saigo and Kilbas [47]. Gorenflo et al. [11] discussed this function presenting recurrence relations and, for a particular case of the parameters, connections with functions of hypergeometric type. Certain properties of fractional calculus operators associated with these generalized Mittag-Leffler functions were also discussed by Saxena and Saigo [48].

For recent applications of the Kilbas and Saigo function we cite the 2009 paper by Orsingher & Polito [43] who discussed the birth-death stochastic processes associated with a time-fractional diffusion equation, the 2010 paper by Kilbas and Repin [25] who discussed an analog of the Tricomi problem for a mixed type equation, and, more recently, the 2013 paper by Hanyga and Seredyńska [15] who discussed a problem associated with the so-called Bloch-Torrey differential equation for an anisotropic time-fractional diffusion equation relevant in the magnetic resonance imaging (MRI), see also [37].

In the literature there are other definitions of multi-index Mittag-Leffler functions, mainly dealt by Kiryakova [32–34] and Luchko [36]. Functions of the Mittag-Leffler type related to fractional calculus have been investigated in the 2006 book by Kilbas et al. [31], and in the forthcoming books by Srivastava [50] and by Gorenflo et al. [10].

3 Fractional differential equations of relaxation type and the Kilbas and Saigo function

In this section we find the solutions of fractional differential equations associated with some relaxation processes in terms of the Kilbas and Saigo function. Our final purpose is to state the completely monotonicity of these solutions for a particular relation involving the parameters entering the governing fractional differential equations[1]. In fact the property of completely monotonicity of the solutions is characteristic of any relaxation process to be considered as a (discrete and/or continuous) superposition of elementary (that is exponential) relaxation processes. In linear viscoelasticity this assumption is usually required, see e.g. [38] and references therein.

Let us consider the following initial-value problem

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} u(t) = -\lambda\, t^\beta u(t), \quad t > 0, \quad u(0) = u_0, \tag{1}$$

where u_0, λ are positive (dimensional) constants, and the (dimensionless) parameters α, β are subjected to the conditions

$$0 < \alpha \leq 1, \quad -\alpha < \beta \leq 1 - \alpha. \tag{2}$$

The above conditions will be conjectured to be sufficient to ensure the existence and complete monotonicity of the solution $u(t)$ for $t \geq 0$. In Eq. (1) the fractional derivative is considered in the Caputo sense, see e.g. [12, 44]. The particular case $\{\alpha = 1, \beta = 0\}$ is associated with the standard relaxation process, whose solution $u(t) = u_0 \exp(-\lambda t)$, is known in the framework of the physical theory of dielectrics as Debye relaxation. With Eq. (1) we intend to generalize the standard relaxation process by introducing a non-constant relaxation coefficient depending on time by a power law and a memory effect due the fractional derivative

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{u}(t')}{(t-t')^\alpha}\, \mathrm{d}t',$$

that for $\alpha = 1$ reduces to the first order derivative $\dot{u}(t)$. We will show that the initial-value provided by Eqs. (1) and (2) is suitable to model some non–Debye (or anomalous) relaxation processes along with the corresponding spectral distributions in frequency. Henceforth, for the sake of convenience, we agree to use non-dimensional quantities by setting $\lambda = 1 = u_0$without loss of generality.

To solve Eq. (1) we proceed as in [30], proposing the ansatz

$$u(t) = \sum_{n=0}^{\infty} (-1)^n c_n(\alpha, \beta) \frac{t^{n(\alpha+\beta)}}{\Gamma[n(\alpha+\beta)+1]}, \tag{3}$$

where $c_n(\alpha, \beta)$ to be determined. Substituting this power series in Eq. (1) and using the relation

[1] Let us recall that a real function $u(t)$ defined for $t \in \mathbb{R}^+$ is said to be completely monotonic (c.m.), if it possesses derivatives $u^{(n)}(t)$ for all $n = 0, 1, 2, 3, \ldots$ and if $(-1)^n u^{(n)}(t) \geq 0$ for all $t > 0$. The limit $u^{(n)}(0^+) = \lim_{t \to 0^+} u^{(n)}(t)$ finite or infinite exists. It is known from the Bernstein theorem that a necessary and sufficient condition that $u(t)$ be c.m. is that

$$u(t) = \int_0^\infty \mathrm{e}^{-rt}\, \mathrm{d}\mu(r),$$

where $\mu(t)$ is non-decreasing and the integral converges for $0 < t < \infty$. In other words $u(t)$ is required to the real Laplace transform of a non negative measure, in particular

$$u(t) = \int_0^\infty \mathrm{e}^{-rt} K(r)\, \mathrm{d}r, \quad K(r) \geq 0,$$

where $K(r)$ is a standard or generalized function known as spectral distribution. For more mathematical details, consult e.g. the survey by Miller and Samko [40].

$$\frac{d^\mu}{dt^\mu}\left\{\frac{t^\xi}{\Gamma(\xi+1)}\right\} = \frac{t^{\xi-\mu}}{\Gamma(\xi-\mu+1)}, \quad t>0, \tag{4}$$

with $0<\mu\leq 1$ and $\xi>0$, we obtain the following recurrence relation

$$c_{n+1}(\alpha,\beta) = \frac{\Gamma[n(\alpha+\beta)+\beta+1]}{\Gamma[n(\alpha+\beta)+1]} c_n(\alpha,\beta). \tag{5}$$

Let us now consider in the complex plane the Mittag-Leffler type function as introduced by Kilbas and Saigo [26, 27] as the power series

$$E_{\alpha,m,\ell}(z) = \sum_{n=0}^{\infty} c_n z^n,\; c_n = \prod_{i=0}^{n-1}\frac{\Gamma[\alpha(im+\ell)+1]}{\Gamma[\alpha(im+\ell+1)+1]}, \tag{6}$$

with $\alpha, m, \ell \in \mathbb{R}$ such that $\alpha>0$, $m>0$ and $\alpha(im+\ell)\neq -1,-2,-3,\ldots$.

In the above assumptions on the parameters this function was proved to be entire of order $\rho = 1/\alpha$ and type $\sigma = m$, that means for $\epsilon>0$

$$|E_{\alpha,m,\ell}(z)| < \exp\left[\left(\frac{1}{m}+\epsilon\right) z^{1/\alpha}\right], \quad z\in\mathbb{C}. \tag{7}$$

In Eq. (6) an empty product is supposed to be equal one, so that $c_0 = 1$.

Then, we recognize that the solution of the initial-value problem Eqs. (1)–(2) is given by

$$\begin{aligned} u(t) &= E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-t^{\alpha+\beta}) \\ &= 1+\sum_{n=1}^{\infty}(-1)^n \prod_{i=0}^{n-1}\frac{\Gamma(i(\alpha+\beta)+\beta+1)}{\Gamma(i(\alpha+\beta)+\alpha+\beta+1)} t^{(\alpha+\beta)n} \end{aligned} \tag{8}$$

with the conditions (2). We find worth to introduce the positive parameter

$$\gamma \equiv \alpha+\beta, \tag{9}$$

so the solution reads

$$\begin{aligned} u(t) &= E_{\alpha,\frac{\gamma}{\alpha},\frac{\gamma-\alpha}{\alpha}}(-t^\gamma) \\ &= 1+\sum_{n=1}^{\infty}(-1)^n \prod_{i=0}^{n-1}\frac{\Gamma(i\gamma+\gamma-\alpha+1)}{\Gamma(i\gamma+\gamma+1)} t^{\gamma n} \end{aligned} \tag{10}$$

with the conditions

$$0<\alpha\leq 1, \quad 0<\gamma\leq 1. \tag{11}$$

In the following we will use the parameters $\{\alpha,\beta\}$ or $\{\alpha,\gamma\}$, according to our convenience.

3.1 Particular cases

Hereafter we recover the solutions corresponding to two particular noteworthy cases of Eq. (10), that is

$$\{0<\alpha<1,\ \beta=0\} \quad \text{and} \quad \{\alpha=1,\ -1<\beta\leq 0\}$$

which are known to be c.m. functions.

For the first particular case we get

$$u(t) = E_{\alpha,1,0}(-t^\alpha) = E_\alpha(-t^\alpha), \quad 0<\alpha<1, \tag{12}$$

where $E_\alpha(\cdot)$ is the classical Mittag-Leffler function which is known to be c.m. for negative argument if $0<\alpha\leq 1$, see e.g. [12].

For the second particular case we show that the solution is associated with the stretched exponential which is a well known c.m. function. In fact we have from Eq. (10),

$$\begin{aligned} u(t) &= E_{1,1+\beta,\beta}(-t^{\beta+1}) \\ &= \sum_{n=0}^{\infty}\prod_{i=0}^{n-1}\frac{\Gamma[i(\beta+1)+\beta+1]}{\Gamma[i(\beta+1)+\beta+2]}\left(-t^{\beta+1}\right)^n \end{aligned} \tag{13}$$

which can be written as

$$u(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(\beta+1)^n} t^{n(\beta+1)} \prod_{i=0}^{n-1}\frac{1}{i+1}. \tag{14}$$

Thus, as the last product is $1/n!$ and using the exponential power series, we finally obtain

$$u(t) = E_{1,1+\beta,\beta}(-t^{\beta+1}) = \exp\left(-\frac{t^{\beta+1}}{\beta+1}\right), \tag{15}$$

which is indeed c.m. for $-1<\beta\leq 0$. Of course such solution can be derived more directly by integrating the ordinary differential equation obtained from Eq. (1) with $\alpha=1$ and $\beta>-1$. Setting $\beta=0$ in this solution we recover

$$u(t) = E_{1,1,0}(-t) = \exp(-t), \tag{16}$$

that is the exponential solution of the standard relaxation equation.

4 Complete monotonicity of the Kilbas and Saigo function

In this section we will *assume* that the conditions on the parameters α and β stated in Eq. (2) ensure the complete monotonicity of $u(t)$ given by Eq. (8). As earlier pointed out, this is equivalent to *assume* that the conditions on α and γ stated in Eq. (11) ensure the complete monotonicity of $u(t)$ given by Eq. (10). For this *assumption* we prefer to manage with the positive parameters α, γ, both of which are required to be less or equal to one.

It is well known that the Bernstein theorem provides a necessary and sufficient condition for the complete monotonicity of a function infinitely differentiable for all $t > 0$. This theorem implies to express our $u(t)$ as the Laplace transform of a non-negative measure, that is

$$u(t) \equiv E_{\alpha,\frac{\gamma}{\alpha},\frac{\gamma-\alpha}{\alpha}}(-t^{\gamma}) = \int_0^{\infty} \mathrm{e}^{-rt} f_{\alpha,\gamma}(r)\,\mathrm{d}r, \tag{17}$$

where $f_{\alpha,\gamma}(r) \geq 0$ in all of $\mathbb{R}^+$ is referred to as the (frequency) spectral distribution. In other words we must determine a function $f_{\alpha,\gamma}(r)$ satisfying the inverse Laplace integral

$$f_{\alpha,\gamma}(r) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \mathrm{e}^{sr} E_{\alpha,\frac{\gamma}{\alpha},\frac{\gamma-\alpha}{\alpha}}(-s^{\gamma})\,\mathrm{d}s \tag{18}$$

with $d = \mathrm{Re}(s) > 0$ and $0 < \alpha \leq 1$ and $0 < \gamma \leq 1$. By using the relation

$$\mathcal{L}^{-1}[s^{-\mu-1}] = r^{\mu}/\Gamma(\mu+1), \quad r > 0, \tag{19}$$

valid in classical sense for $\mu > -1$ and in generalises sense also for $\mu \leq -1$, see e.g. [7], we obtain by inverting term by term the series expansion of $E_{\alpha,\frac{\gamma}{\alpha},\frac{\gamma-\alpha}{\alpha}}(-s^{\gamma})$ in Eq. (18) the required spectral distribution for $r > 0$,

$$f_{\alpha,\gamma}(r) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(-\gamma n)} \prod_{j=0}^{n-1} \frac{\Gamma(\gamma j + \gamma - \alpha + 1)}{\Gamma(\gamma j + \gamma + 1)} \left(\frac{1}{r}\right)^{\gamma n + 1}. \tag{20}$$

For details we refer to Appendix 1.

We can arrive at Eq. (20) in an alternative way by recognizing that the Laplace transform of $u(t)$ is the Stieltjes transform (that is the iterated Laplace transform) of the spectral distribution. As a consequence $f_{\alpha,\gamma}(r)$ can be obtained by the Titchmarsch formula for the inversion of the Stieltjes transform. See for details Appendix 2.

We note that the series representation in negative powers of r provides a limitation to the actual determination of the spectral distribution $K(r)$ for all $r \geq 0$ because we expect a numerical instability for r sufficiently close to the origin. However, in some particular cases, see below, it is possible to sum exactly the series by an analytical expression valid for all $r > 0$.

4.1 Particular cases

Henceforth we derive the spectral distribution for the particular cases of the parameters α and $\gamma = \alpha + \beta$ considered in Subsection 3.1.

As a first particular case, we discuss the case $\gamma = \alpha$, i.e. $0 < \alpha < 1$ and $\beta = 0$ whose solution is given by Eq. (12), $u(t) = u_0\,E_{\alpha,1,0}(-t^{\alpha}) = u_0\,E_{\alpha}(-t^{\alpha})$. Then, setting $\gamma = \alpha$ in Eq. (20), we get for $r > 0$

$$f_{\alpha,\alpha}(r) = \frac{1}{r} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(-\alpha n)} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha j + 1)}{\Gamma(\alpha j + \alpha + 1)} \left(\frac{1}{r}\right)^{\alpha n}.$$

Using the relation

$$\prod_{j=0}^{n-1} \frac{\Gamma(\alpha j + 1)}{\Gamma(\alpha j + \alpha + 1)} = \frac{1}{\Gamma(\alpha n + 1)}$$

we can write for $r > 0$

$$f_{\alpha,\alpha}(r) = \frac{1}{r} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(-\alpha n)} \frac{(1/r)^{\alpha n}}{\Gamma(\alpha n + 1)},$$

Using the reflection formula for the gamma function the above equation can be written as

$$f_{\alpha,\alpha}(r) = \frac{1}{\pi r} \sum_{n=1}^{\infty} (-1)^{n-1} \sin(\pi\alpha n)(1/r)^{\alpha n},\ r > 0. \tag{21}$$

To evaluate this sum we use the geometric series getting, see Appendix 3,

$$f_{\alpha,\alpha}(r) = \frac{1}{\pi r} \frac{\sin(\pi\alpha)}{r^{\alpha} + 2\cos(\pi\alpha) + r^{-\alpha}}, \quad r > 0, \tag{22}$$

which is always non-negative for $r > 0$. This result concerning the spectral distribution of the classical Mittag-Leffler function is well known and was discussed in detail by Gorenflo and Mainardi [12].

A second particular case is concerning the stretched exponential solution Eq. (15) obtained for $\alpha = 1$ and $0 < \gamma < 1$ (i.e. $-1 < \beta < 0$). In this case the spectral distribution is known to be the unilateral extremal

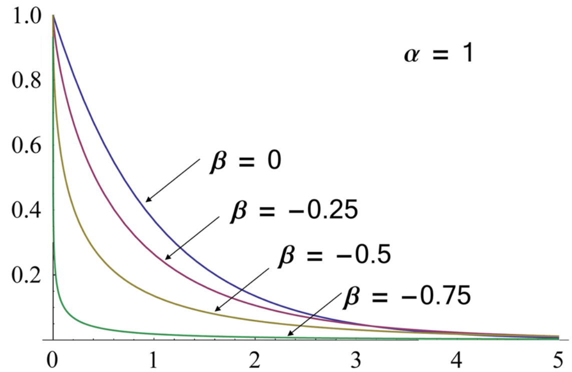

Fig. 1 Plots of $u(t)$ for $\alpha = 1$ for selected values of $\beta \in (-\alpha, 1-\alpha]$ in the time range $0 \le t \le 5$

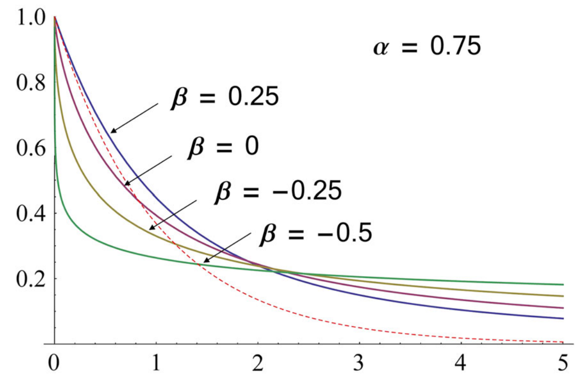

Fig. 2 Plots of $u(t)$ for $\alpha = 0.75$ for selected values of $\beta \in (-\alpha, 1-\alpha]$ in the time range $0 \le t \le 5$

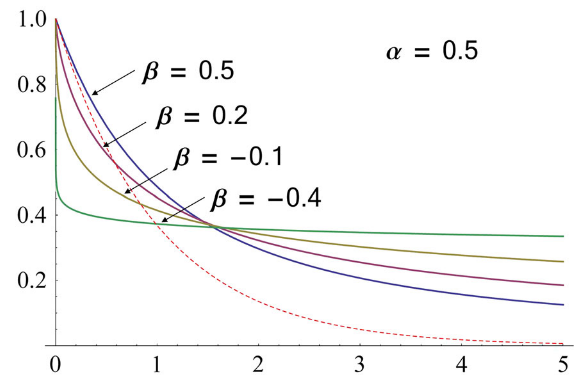

Fig. 3 Plots of $u(t)$ for $\alpha = 0.5$ for selected values of $\beta \in (-\alpha, 1-\alpha]$ in the time range $0 \le t \le 5$

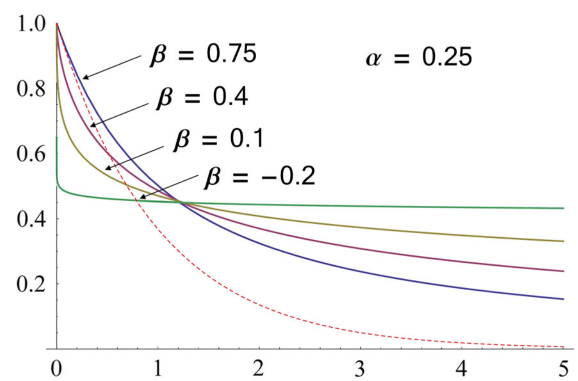

Fig. 4 Plots of the $u(t)$ for $\alpha = 0.25$ for selected values of $\beta \in (-\alpha, 1-\alpha]$ in the time range $0 \le t \le 5$

Lévy density of order $\gamma = 1 + \beta$ expressed in terms of the Wright function (of the second kind), see e.g. Appendix F in [38]

$$\begin{aligned} f_{1,\gamma}(r) &= \frac{1}{r}\, W_{-\gamma,0}(-r^{-\gamma}/\gamma) \\ &= \frac{1}{r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{(r^{-\gamma}/\gamma)^n}{\Gamma(-\gamma n)},\; r > 0. \end{aligned} \tag{23}$$

The details of this derivation are found in Appendix 4.

5 Numerical results

Here we discuss the numerical results concerning the solutions and the spectral distributions previously obtained in analytical closed-form by exhibiting the corresponding plots for selected values of the parameters.

At first we present the plots referring to the solution given by Eq. (8) for $\alpha = 1, 0.75, 0.50, 0.25 \in (0, 1]$, respectively in Figs. 1, 2, 3, and 4. For each value of α we have selected a few values of β such that $-\alpha < \beta \le 1 - \alpha$ as required by Eq. (2) so that $0 < \gamma \equiv \alpha + \beta \le 1$ as required by Eq. (11).

We note that only in the case $\alpha = 1$ we get an exponential-like decay whereas for $0 < \alpha < 1$ the decay is of power law type, that is much slower than the standard exponential decay pointed out with a dotted line. The evaluation of the general decay law in terms of the order parameters α and β (or α and $\gamma = \alpha + \beta$) will be left to a next paper.

Then, we would like to exhibit the plots for the spectral distributions corresponding to all the above cases. However, because of possible numerical instabilities for $r \to 0$ we limit ourselves to those cases where it is possible to plot the corresponding spectral distributions also for values of r close to zero.

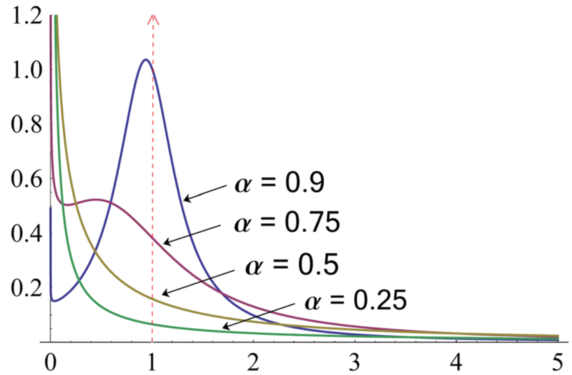

Fig. 5 Plots of the spectral distribution for $\alpha = \gamma = 0.25, 0.50, 0.75, 0.90$ ($\beta = 0$) in the range $0 \le r \le 5$

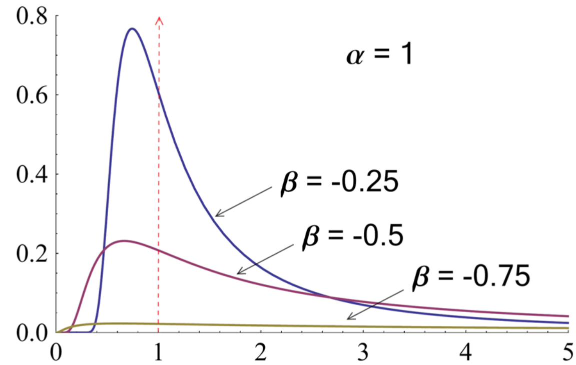

Fig. 6 Plots of the spectral distribution for $\alpha = 1$ and $\beta = -0.25, -0.50, -0.75$ in the range $0 \le r \le 5$

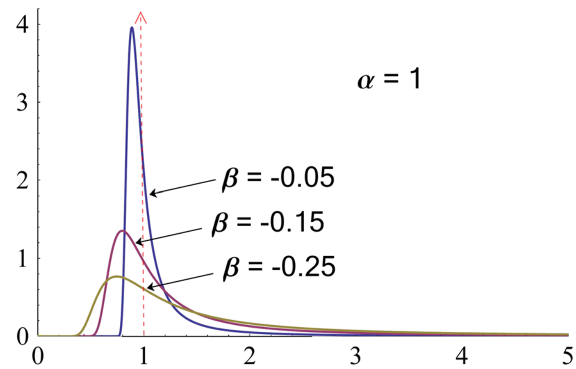

Fig. 7 Plots of the spectral distribution for $\alpha = 1$ and $\beta = -0.05, -0.15, -0.25$ in the range $0 \le r \le 5$

At first we consider the case $\{0 < \alpha < 1, \beta = 0\}$ so $\gamma = \alpha$ for which the series of $f_{\alpha,\alpha}(r)$ can be summed to give the analytical formula Eq. (22), see also Eq.(39). The spectral distribution is illustrated in Fig. 5 for $\alpha = 0.25, 0.50, 0.75, 0.90$. For $\alpha = 1$ we recover the delta function $\delta(r-1)$ centred in $r = 1$.

Then, we consider the case $\{\alpha = 1, -1 < \beta \le 0\}$ for which we have provided in Appendix 4, two different representations for the spectral distribution $f_{1,\gamma}(r)$ ($\gamma = 1 + \beta$), see the power series in Eq. (43) and the integral in Eq. (47). We note that, from numerical view point, for small values of r we must take profit of the asymptotic representation valid for $r \to 0$ in Eq. (49).

In Figs. 6, 7 we show the spectral distributions for a few values of γ, that is for $\{\beta = -0.25, -0.50. -0.75\}$ and $\beta = -0.05, -0.15, -0.25\}$, respectively. Again for $\alpha = \gamma = 1$ ($\beta = 0$) we recover the delta function centred in $r = 1$.

6 Concluding remarks

In this paper we have analysed some generalized models of relaxation processes which exhibit memory effects and a time varying coefficient. Indeed we have introduced for the field variable a fractional ordinary differential equation of order α with a coefficient varying in time as a power law with exponent β. The resulting process depending on the two parameters $\{\alpha, \beta\}$ has been dealt with the Kilbas and Saigo function (of the Mittag-Leffler type) after having presented an overview on this transcendental function. Because any relaxation process would be expressed as a continuous or discrete superposition of elementary exponential processes, the field variable of our process would result a completely monotone function with a non-negative spectral distribution of frequencies. By a conjecture, we have stated that the conditions on the two parameters α, β which are expected to ensure the complete monotonicity the solution expressed by a Kilbas and Saigo function. In this paper the corresponding spectral distribution has been given in terms of a negative power series of the frequency. Two noteworthy one-parameter processes, namely those governed by the standard Mittag-Leffler function and by the stretched exponential, have been recovered as particular cases. For some study-cases we have presented numerical results with illustrative plots for the field variable and for the corresponding spectral distribution. Although we were not able to provide a rigorous mathematical proof, our conjecture for the complete monotonicity has been confirmed in our numerical results. We hope that our results can be adopted when the field variable is the response function associated with non–Debye relaxation processes found e.g. in dielectrics.

For reader's convenience, the calculations not contained in the text have been enclosed with details in four appendices. We point out the limitations of our method in deriving the spectral distributions as a power series that could provide numerical instabilities for very small frequencies. To overcome this trouble it would be necessary to derive a matching with a general asymptotic representation for small frequencies that could be obtained from the asymptotic behaviour of the response function for large times, in view of the Tauberian theorems for Laplace transforms. For a next paper we leave all the questions not totally solved with the present analysis.

We finally note that a novel approach to non–Debye relaxation has been considered in a recent paper by Garra et al. [9]. Being based on a different differential equation of fractional order with a non-constant coefficients, their approach can be considered complementary to ours.

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Appendix 1: The spectral distribution as inverse Laplace transform

In this Appendix we will obtain $f_{\alpha,\beta}(r)$ as provided by Eq. (20). We know from the Bernstein theorem that $f_{\alpha,\gamma}(r)$ would be associated with the Laplace transform

$$u(t) \equiv E_{\alpha,\frac{\gamma}{\alpha},\frac{\gamma-\alpha}{\alpha}}(-t^{\gamma}) = \int_0^{\infty} \mathrm{e}^{-rt} f_{\alpha,\gamma}(r)\,\mathrm{d}r, \tag{24}$$

where we have recalled the expression of $u(t)$ from Eq. (10).

The corresponding inverse Laplace transform furnishes

$$f_{\alpha,\gamma}(r) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \mathrm{e}^{rt} u(t)\,\mathrm{d}t, \quad d > 0, \tag{25}$$

with $r > 0$. Inserting $u(t)$ as provided by Eq. (10) and rearranging we get

$$f_{\alpha,\gamma}(r) = \sum_{n=1}^{\infty}(-1)^n \prod_{i=0}^{n-1} \frac{\Gamma(i\gamma+\gamma-\alpha+1)}{\Gamma(i\gamma+\gamma+1)} \times \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \mathrm{e}^{rt} t^{\gamma n}\,\mathrm{d}t \tag{26}$$

where we have neglected the constant 1 in the inversion formula because its singular contribution (of δ type) is vanishing for $r > 0$. Using the relation

$$\mathcal{L}^{-1}[t^{\gamma n}] = \frac{r^{-n\gamma-1}}{\Gamma(-n\gamma)}, \quad t, r > 0, \tag{27}$$

and substituting this result in the last equation we have

$$f_{\alpha,\gamma}(r) = \sum_{n=1}^{\infty} A^n_{\alpha,\gamma} \frac{(-1)^n}{\Gamma(-n\gamma)}, \tag{28}$$

where we have introduced

$$A^n_{\alpha,\gamma}(r) = \prod_{i=0}^{n-1} \frac{1}{r^{n\gamma+1}} \frac{\Gamma(i\gamma+\gamma-\alpha+1)}{\Gamma(i\gamma+\gamma+1)} \tag{29}$$

which is always positive. Using the reflection formula for the gamma function

$$\frac{1}{\Gamma(-n\gamma)} = -\frac{1}{\pi}\Gamma(1+n\gamma)\sin(\pi n\gamma) \tag{30}$$

we can write

$$f_{\alpha,\gamma}(r) = \frac{1}{\pi}\sum_{n=1}^{\infty} B^n_{\alpha,\gamma}(r)(-1)^{n-1}\sin(\pi n\gamma), \tag{31}$$

where $A^n_{\alpha,\gamma}(r)\Gamma(1+n\gamma) = B^n_{\alpha,\gamma}(r) > 0$.

Appendix 2: The spectral distribution as inverse Stieltjes transform

In this Appendix we obtain the spectral distribution as inverse Stieltjes transform. First we calculate the Laplace transform of $u(t)$ as given in Eq. (10),

$$\mathcal{L}[u(t)] = \frac{1}{s} + \sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{\Gamma(i\gamma+\gamma-\alpha+1)}{\Gamma(i\gamma+\gamma+1)} \times (-1)^n \int_0^{\infty} e^{-st} t^{n\gamma}\, dt. \quad (32)$$

Evaluating the last integral we can write

$$\widetilde{u}(s) = \frac{1}{s} + \frac{1}{s}\sum_{n=1}^{\infty}(-1)^n \frac{\Gamma(n\gamma+1)}{s^{n\gamma}}\prod_{i=0}^{n-1}\frac{\Gamma(i\gamma+\gamma-\alpha+1)}{\Gamma(i\gamma+\gamma+1)}, \quad (33)$$

where $\widetilde{u}(s) = \mathcal{L}[u(t)]$. Thus, to obtain $f_{\alpha,\beta}(r)$ by means of the inverse Stieltjes transform we must use the Titchmarsh formula, see *e.g.* Titchmarsh [51], pp 317–319, Widder [55], pp. 339–341, and Gross [13], namely,

$$f_{\alpha,\gamma}(r) = \mp\frac{1}{\pi}\mathrm{Im}\left\{\widetilde{u}(s)|_{s=r\,e^{\pm i\pi}}\right\}, \quad (34)$$

with $\mathrm{Im}\{\widetilde{u}(s)\}$ denoting the imaginary part of $\widetilde{u}(s)$. Writing the exponential in terms of $\cos(\pi n\gamma)$ and $\sin(\pi n\gamma)$ and taking the imaginary part we have

$$\mathrm{Im}\left\{\widetilde{u}(s)|_{s=r\,e^{\pm i\pi}}\right\} = -\frac{1}{\pi}\frac{\sin(\pi n\gamma)}{r^{n\gamma+1}} \times \prod_{i=0}^{n-1}\frac{\Gamma(i\gamma+\gamma-\alpha+1)}{\Gamma(i\gamma+\gamma+1)} \quad (35)$$

We then use the reflection formula for the gamma function to get

$$f_{\alpha,\gamma}(r) = \frac{1}{r}\sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{(-1)^n}{\Gamma(-n\gamma)}\frac{\Gamma(i\gamma+\gamma-\alpha+1)}{\Gamma(i\gamma+\gamma+1)}\frac{1}{r^{n\gamma}} \quad (36)$$

valid for $0<\gamma\leq 1$.

Appendix 3: The spectral distribution of the Mittag-Leffler function

In this Appendix we explicitly derive from Eq. (21) the formula of the spectral distribution of the Mittag-Leffler function as provided by Eq. (22). To see this result we introduce $2i\sin(\pi\alpha n) = e^{i\pi\alpha n} - e^{-i\pi\alpha n}$ in Eq. (21), i.e.,

$$f_{\alpha,\alpha}(r) = -\frac{1}{2\pi i r}\sum_{n=1}^{\infty}(-1)^n\left(e^{i\pi\alpha n} - e^{-i\pi\alpha n}\right)r^{-\alpha n}. \quad (37)$$

Separating in two sums, we rewrite the above equation in the form

$$f_{\alpha,\alpha}(r) = -\frac{1}{2\pi i r}\sum_{n=0}^{\infty}\left(-r^{-\alpha}e^{i\pi\alpha}\right)^n + \frac{1}{2\pi i r}\sum_{n=0}^{\infty}\left(-r^{-\alpha}e^{-i\pi\alpha}\right)^n. \quad (38)$$

We note that the sum index beginning at $n=0$ allows us to write the two geometric series in terms of their sums valid for $|r^{\alpha}|>1$ as follows

$$f_{\alpha,\alpha}(r) = -\frac{1}{2\pi i r}\left\{\frac{1}{1+r^{-\alpha}e^{i\pi\alpha}}\right\} + \frac{1}{2\pi i r}\left\{\frac{1}{1+r^{-\alpha}e^{-i\pi\alpha}}\right\}$$

which can be rewritten in the form

$$f_{\alpha,\alpha} = \frac{1}{2\pi i r}\frac{(1+r^{-\alpha}e^{i\pi\alpha})-(1+r^{-\alpha}e^{-i\pi\alpha})}{(1+r^{-\alpha}e^{i\pi\alpha})(1+r^{-\alpha}e^{-i\pi\alpha})} = \frac{1}{2\pi i r}\frac{r^{-\alpha}[2i\sin(\pi\alpha)]}{1+r^{-\alpha}\cdot 2\cos(\pi\alpha)+r^{-2\alpha}},$$

so that

$$f_{\alpha,\alpha} = \frac{1}{\pi r}\frac{\sin(\pi\alpha)}{r^{\alpha}+2\cos(\pi\alpha)+r^{-\alpha}}, \quad (39)$$

which is the result obtained in Eq. (22). Note that this result is valid for all $r>0$ even if obtained from the sum of two geometric series divergent for $0<r<1$. In particular, for $r=1$ Eq. (39) provides the result Eq. (40).

Appendix 4: The spectral distribution of the stretched exponential

In this Appendix we recover, as a particular case, the spectral distribution $f_{1,\gamma}(r)$ of the stretched exponential

$$u(t) = E_{1,\gamma,\gamma-1}(-t^{\gamma}) = E_1(-t^{\gamma}/\gamma) = \exp(-t^{\gamma}/\gamma), \quad (40)$$

where we recall $0<\gamma=1+\beta<1$. The limiting case $\gamma=1$ $(\beta=0)$ corresponding to the exponential $\exp(-t)$ is excluded because the spectral distribution degenerates into the Dirac delta function $\delta(r-1)$.

Then, putting $\alpha=1$ in Eq. (36) we get

$$f_{1,\gamma}(r) = \frac{1}{r}\sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{(-1)^n}{\Gamma(-n\gamma)}\frac{\Gamma(i\gamma+\gamma)}{\Gamma(i\gamma+\gamma+1)}\frac{1}{r^{n\gamma}}, \quad (41)$$

valid for $0<\gamma\leq 1$. Evaluating the product, we have

$$\prod_{i=0}^{n-1}\frac{\Gamma(i\gamma+\gamma)}{(i\gamma+\gamma)\Gamma(i\gamma+\gamma)}=\prod_{i=0}^{n-1}\frac{1}{i\gamma+\gamma}=\frac{1}{\gamma^n}\frac{1}{n!}. \tag{42}$$

Thus, we obtain the spectral distribution given by

$$f_{1,\gamma}(r)=\frac{1}{r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\Gamma(-n\gamma)}\left(\frac{r^{-\gamma}}{\gamma}\right)^n,\quad r>0, \tag{43}$$

in agreement with Eq. (23).

It is instructive to consider the particular case $\gamma=1/2$, for which we get

$$f_{1,\frac{1}{2}}(r)=\frac{1}{r}\sum_{n=1}^{\infty}\frac{(-2)^n}{n!\Gamma(-n/2)}r^{-n/2}. \tag{44}$$

Using the reflection formula for the gamma function and taking $n\to 2n+1$ we have

$$f_{1,\frac{1}{2}}(r)=\frac{1}{\pi r}\sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\frac{\Gamma(n+3/2)}{(2n+1)!}r^{-n-1/2}. \tag{45}$$

Finally, by means of the duplication formula for the gamma function and simplifying we get

$$f_{1,\frac{1}{2}}(r)=\frac{1}{\sqrt{\pi}r^{3/2}}\sum_{n=0}^{\infty}(-1)^n\frac{r^{-n}}{n!}=\frac{1}{\sqrt{\pi}}r^{-3/2}\,\mathrm{e}^{-1/r}. \tag{46}$$

This result is well known from the standard tables of Laplace transforms being the inverse of $\exp(-2s^{1/2})$, see e.g. [7], No (49), p. 320.

We recognize that the power series in Eq. (43), even if mathematically convergent for all $r>0$ is not suitable to represent numerically the function for sufficiently small r. In fact in the complex plane the high transcendental function $f_{1,\gamma}(z)$, being a Wright function of the second kind (see Appendix F in [38]), exhibits an essential singularity in $z=0$, as it can be understood in the particular case $\gamma=1/2$ in Eq. (46). In other words, we expect for sufficiently small values of r a very strong rate of change in the spectral distribution. So we look for an integral representation alternative to the series representation Eq. (43) that may be more suitable for numerical computation.

To this end, first of all, we consider the integral representation for the reciprocal of gamma function

$$\frac{1}{\Gamma(-\gamma k)}=\frac{i}{2\pi}\int_{\mathrm{Ha}}\mathrm{e}^{-t}(-t)^{\gamma k}\,\mathrm{d}t,$$

where Ha is the Hankel contour in the complex plane. Substituting this expression in Eq. (43) and rearranging we get

$$f_{1,\gamma}(r)=\frac{i}{2\pi r}\int_{\mathrm{Ha}}\mathrm{d}t\,\mathrm{e}^{-t}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{r^{-\gamma}}{\gamma}\right)^k\left(\mathrm{e}^{-i\pi\gamma}t^{\gamma}\right)^k\right\}.$$

Using the definition of the exponential function, we can write

$$f_{1,\gamma}(r)=\frac{i}{2\pi r}\int_{\mathrm{Ha}}\mathrm{d}t\exp\left[-t-\frac{\mathrm{e}^{i\pi\gamma}}{\gamma}\left(\frac{t}{r}\right)^{\gamma}\right],$$

or in the following form

$$f_{1,\gamma}(r)=\frac{1}{\pi r}\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-t}\exp\left[-\frac{\cos\pi\gamma}{\gamma}\left(\frac{t}{r}\right)^{\gamma}\right]\times\sin\left[\frac{\sin\pi\gamma}{\gamma}\left(\frac{t}{r}\right)^{\gamma}\right], \tag{47}$$

which is the required integral representation associated with the Eq. (43).

As an example, we recover the result obtained in Eq. (46). In this particular case, we consider $\gamma=1/2$. Taking $\gamma=1/2$ in Eq. (47) we get

$$f_{1,\frac{1}{2}}(r)=\frac{1}{\pi r}\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-t}\sin\left[2\left(\frac{t}{r}\right)^{\frac{1}{2}}\right].$$

Introducing the variable $t=u^2$ and using the relation

$$\int_0^{\infty}\mathrm{e}^{-\beta x^2}\cos bx\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(-\frac{b^2}{4\beta}\right),$$

we have

$$f_{1,\frac{1}{2}}(r)=\frac{2}{\pi r}\sqrt{\frac{\pi}{r}}\frac{1}{2}\,\mathrm{e}^{-1/r}=\frac{1}{\sqrt{\pi}}r^{-3/2}\,\mathrm{e}^{-1/r}, \tag{48}$$

which is the same result as obtained in Eq. (46).

However, from a numerical view point, we note that we can use the integral representation in Eq. (47) for the plot of $f_{1,\gamma}(r)$ until the point it starts to oscillate very rapidly. In the integral representation we have a $\cos(\pi\gamma)$ in the argument of an exponential, and when $\gamma>1/2$ this causes a problem for numerical evaluation of the integral when in association with the increase of the oscillation in the term with the sin term. But in this case we must use the asymptotic representation of the M-Wright function (see Eq. (F.20) in Appendix F in [38]) since we have as $r \to 0$,

$$f_{1,\gamma}(r) = r^{-1-\gamma} M_\gamma(r^{-\gamma}/\gamma) \sim \frac{r^{-\frac{1-\gamma/2}{1-\gamma}}}{\sqrt{2\pi(1-\gamma)}} \exp\left[-\frac{1-\gamma}{\gamma} r^{-\frac{\gamma}{1-\gamma}}\right]. \tag{49}$$

For $\gamma = 1/2$ the asymptotic formula in Eq. (49) provides the exact result found in Eqs. (46) and (48).

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