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ORIGINAL ARTICLE

On computation of optimal strategies in oligopolistic markets respecting the cost of change

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Abstract

The paper deals with a class of parameterized equilibrium problems, where the objectives of the players do possess nonsmooth terms. The respective Nash equilibria can be characterized via a parameter-dependent variational inequality of the second kind, whose Lipschitzian stability, under appropriate conditions, is established. This theory is then applied to evolution of an oligopolistic market in which the firms adapt their production strategies to changing input costs, while each change of the production is associated with some "costs of change". We examine both the Cournot-Nash equilibria as well as the two-level case, when one firm decides to take over the role of the Leader (Stackelberg equilibrium). The impact of costs of change is illustrated by academic examples.

Keywords Generalized equation · Equilibrium · Cost of Change

Mathematics Subject Classification 90C33 · 91B52 · 49J40 · 90C31

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1 Introduction

Consider an oligopolistic market, where the data of the production cost functions and constraints of each producer/firm are available to all his rivals. In such a case each producer can compute his optimal non-cooperative Cournot-Nash strategy by solving the corresponding variational inequality, see [Murphy et al. (1982) and Outrata et al. (1998)]. It may happen, however, that in the course of time some external parameters change, e.g., the prices of the inputs or the parameters of the inverse demand function describing the behavior of the customers. In such a case, the strategies should be adjusted but, as thoroughly analyzed in Flåm (2020), each change is generally associated with some expenses, called *costs of change.* Thus, given a certain *initial strategy profile* (productions of all firms), we face then a different equilibrium model, in which the costs of change enter the objectives of some (or all) producers. Since these costs are typically nonsmooth, the respective variational inequality, describing the new non-cooperative equilibrium, becomes substantially more complicated, both from the theoretical as well as from the numerical point of view. One can imagine that such updates of strategies are performed repetitively. This leads to a discrete-time evolution process, where the firms respond to changing conditions by repetitive solution of the mentioned rather complicated variational inequality (with updated data). As discussed in (Outrata et al. 1998, Chapter 12), it may also happen that one of the producers, having an advantage over the others, takes over the role of a Leader and switches to the *Stackelberg strategy*, whereas the remaining firms continue to play non-cooperatively with each other on the lower level as Followers. In this case, our discrete-time evolution process amounts to repetitive solution of a *bilevel game* in which the players possess nonsmooth objectives. Further, it is interesting to note that the above described model has, in case of positively homogeneous costs of change, a similar structure as some infinite-dimensional variational systems used in continuum mechanics to model a class of *rate-independent processes* cf., e.g., (Mielke and Roubíček (2015)) or (Frost et al. (2019)).

The plan of the paper is as follows. In the preliminary Sect. 2 we collect the necessary background from variational analysis. Section 3 consists of two parts. In the first one we introduce a general parameter-dependent non-cooperative equilibrium problem which is later used for modeling of the considered oligopolistic market. By employing standard arguments, existence of the respective solutions (equilibria) is shown. In the second part we then consider a parameter-dependent variational system which encompasses the mentioned equilibrium problem and is amenable to advanced tools of variational analysis. In this way one obtains a useful stability result concerning the respective *solution map*, whose special variants are presented in the Appendix. This result is used in the sequel but it is important also for its own sake.

Thereafter, in Sect. 4, this equilibrium problem is specialized to a form, corresponding to the oligopolistic market model from Murphy et al. (1982). In this case, the solution map is indeed single-valued and locally Lipschitzian. In Sect. 5 we then consider a modification of the 5-firm example from Murphy et al. (1982) with the aim to illustrate the role of costs of change and to describe a possible numerical approach to the computation of the respective equilibria. Whereas Section 5.1 deals with the noncooperative Cournot-Nash equilibrium, Section 5.2 concerns the situation, when one of the firms prefers to apply the Stackelberg strategy. In both cases our main numerical tool is the forward-backward splitting method described in Facchinei and Pang (2003) which may easily be adapted to the considered type of problems.

The following notation is employed. For a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, gph F signifies the graph of F, δ_A is the indicatory function of a set A and $\overline{\mathbb{R}} = R \cup \{+\infty\}$ is the extended real line. $\mathbb{B}$ stands for the unit ball and, for a cone K, K° denotes its (negative) polar. Finally, $\xrightarrow{A}$ means convergence within a set A.

2 Background from variational analysis

Throughout the whole paper, we will make an extensive use of the following basic notions of modern variational analysis.

Definition 1 Let A be a closed set in $\mathbb{R}^n$ and $\bar{x} \in A$. Then

$$T_A(\bar{x}) := \operatorname*{Limsup}_{t \searrow 0} \frac{A - \bar{x}}{t} = \{h \in \mathbb{R}^n | \exists\, h_i \to h, t_i \searrow 0 \text{ such that } \bar{x} + t_i h_i \in A\ \forall i\}$$

is the *tangent (contingent, Bouligand)* cone to A at $\bar{x}$,

$$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ$$

is the *regular (Fréchet) normal cone* to A at $\bar{x}$, and

$$N_A(\bar{x}) := \operatorname*{Limsup}_{x \xrightarrow{A} \bar{x}} \widehat{N}_A(x) = \{x^* \in \mathbb{R}^n | \exists\, x_i \xrightarrow{A} \bar{x}, x_i^* \in \widehat{N}_A(x_i) \text{ such that } x_i^* \to x^*\}$$

is the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$.

In this definition "Limsup" stands for the Painlevé-Kuratowski *outer set limit*. If A is convex, then $\widehat{N}_A(\bar{x}) = N_A(\bar{x})$ amounts to the classical normal cone in the sense of convex analysis and we write $N_A(\bar{x})$.

The above listed cones enable us to describe the local behavior of set-valued maps via various generalized derivatives. Consider a closed-graph multifunction F and the point $(\bar{x}, \bar{y}) \in$ gph F.

Definition 2 (i) The multifunction $DF(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, defined by

$$DF(\bar{x}, \bar{y})(u) := \{v \in \mathbb{R}^m | (u, v) \in T_{\text{gph}\, F}(\bar{x}, \bar{y})\}, u \in \mathbb{R}^n,$$

is called the *graphical derivative* of F at $(\bar{x}, \bar{y})$;

(ii) The multifunction $D^*F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$D^*F(\bar{x}, \bar{y})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\text{gph}\, F}(\bar{x}, \bar{y})\}, v^* \in \mathbb{R}^m,$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{x}, \bar{y})$.

Next we turn our attention to a proper convex, lower-semicontinuous (lsc) function $q : \mathbb{R}^n \to \overline{\mathbb{R}}$. Given an $\bar{x} \in \operatorname{dom} q$, by $\partial q(\bar{x})$ we denote the classical Moreau-Rockafellar *subdifferential* of q at $\bar{x}$. In this case, for the *subderivative* function $dq(\bar{x}) : \mathbb{R}^n \to \overline{\mathbb{R}}$ (Rockafellar and Wets 1998, Definition 8.1) it holds that

$$dq(\bar{x})(w) = q'(\bar{x}; w) := \lim_{\tau \searrow 0} \frac{q(\bar{x} + \tau w) - q(\bar{x})}{\tau} \text{ for all } \quad w \in \mathbb{R}^n.$$

In Section 3 we will employ also second-order subdifferentials and second-order subderivatives of q.

Definition 3 Let $\bar{v} \in \partial q(\bar{x})$. The multifunction $\partial^2 q(\bar{x}, \bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial^2 q(\bar{x}, \bar{v})(v^*) := D^* \partial q(\bar{x}, \bar{v})(v^*), \quad v^* \in \mathbb{R}^n,$$

is called the *second-order subdifferential* of q at $(\bar{x}, \bar{v})$.

If q is separable, i.e., $q(x) = \sum_{i=1}^n q_i(x_i)$ with some proper convex, lsc functions $q_i : \mathbb{R} \to \overline{\mathbb{R}}$, i=1, 2, …, n, then

$$\partial^2 q(\bar{x}, \bar{v})(v^*) = \begin{bmatrix} \partial^2 q_1(\bar{x}_1, \bar{v}_1)(v_1^*) \\ \vdots \\ \partial^2 q_n(\bar{x}_n, \bar{v}_n)(v_n^*) \end{bmatrix},$$

where $\bar{v}_i$, v_i^* are the ith components of the vectors, $\bar{v}$, v^*, respectively.

Concerning second-order subderivatives (Rockafellar and Wets 1998, Definition 13.3), we confine ourselves to the case when q is, in addition, *piecewise linear-quadratic*. This means that $\operatorname{dom} q$ can be represented as the union of finitely many polyhedral sets, relative to each of which $q(x)$ is given in the form $\frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \alpha$ for some scalar $\alpha \in \mathbb{R}$, vector $a \in \mathbb{R}^n$ and a symmetric $[n \times n]$ matrix A, cf. (Rockafellar and Wets 1998, Definition 10.20).

In this particular case it has been proved in (Rockafellar and Wets 1998, Proposition 13.9) that, with $\bar{v} \in \partial q(\bar{x})$ and $w \in \mathbb{R}^n$ the second-order subderivative $d^2 q(\bar{x}|\bar{v})$ is proper convex and piecewise linear quadratic and

$$d^2 q(\bar{x}|\bar{v})(w) = q''(\bar{x}; w) + \delta_{K(\bar{x}, \bar{v})}(w), \tag{1}$$

where

$$q''(\bar{x}; w) := \lim_{\tau \searrow 0} \frac{q(\bar{x} + \tau w) - q(\bar{x}) - \tau q'(\bar{x}; w)}{\frac{1}{2}\tau^2}$$

is the *one-sided second directional derivative* of q at $\bar{x}$ in direction w and $K(\bar{x}, \bar{v}) := \{w | q'(\bar{x}; w) = \langle \bar{v}, w\rangle\}$. For a general theory of second-order subderivatives (without our restrictive requirements) the interested reader is referred to (Rockafellar and Wets 1998, Chapter 13 B).

We proceed now to the definitions of two important Lipschitzian stability notions for multifunctions which will be extensively employed in the sequel.

Definition 4 Consider a multifunction $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a point $(\bar{u}, \bar{v}) \in \operatorname{gph} S$.

(i) S is said to have the *Aubin property* around $(\bar{u}, \bar{v})$, provided there are neighborhoods $\mathcal{U}$ of $\bar{u}$, $\mathcal{V}$ of $\bar{v}$ along with a constant $\eta \geq 0$ such that

$$S(u_1) \cap \mathcal{V} \subset S(u_2) + \eta \|u_1 - u_2\| \mathbb{B} \text{ for all } u_1, u_2 \in \mathcal{U}.$$

(ii) We say that S has a *single-valued and Lipschitzian localization* around $(\bar{u}, \bar{v})$, provided there are neighborhoods $\mathcal{U}$ of $\bar{u}$, $\mathcal{V}$ of $\bar{v}$ and a Lipschitzian mapping $s : \mathcal{U} \to \mathbb{R}^n$ such that $s(\bar{u}) = \bar{v}$ and

$$S(u) \cap \mathcal{V} = \{s(u)\} \text{ for all } u \in \mathcal{U}.$$

Further important stability notions can be found, e.g., in Dontchev and Rockafellar (2014).

Finally we recall an important notion from the theory of monotone operators. Given a maximal monotone operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a positive real c, the mapping $J_{cT} := (I + cT)^{-1}$ is called the *resolvent* of T (with constant c). This mapping is maximal monotone too, and single-valued over $\mathbb{R}^n$. For other properties of the resolvent we refer to (Rockafellar and Wets 1998, Chapter 12).

3 General equilibrium model: existence and stability

Consider a non-cooperative game of l players, each of which solves the optimization problem

$$\begin{array}{ll} \text{minimize} & f_i(p, x_i, x_{-i}) + q_i(x_i) \\ \text{subject to} & \\ & x_i \in A_i, \end{array} \tag{2}$$

$i = 1, 2, \ldots, l$. In (2), $x_i \in \mathbb{R}^n$ is the *strategy* of the ith player,

$$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l) \in (\mathbb{R}^n)^{l-1}$$

is the *strategy profile* of the remaining players and $p \in \mathbb{R}^m$ is a parameter, common for all players. Further, the functions

$$f_i : \mathbb{R}^m \times (\mathbb{R}^n)^l \to \mathbb{R} \quad \text{and} \quad q_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, 2, \ldots, l,$$

are continuously differentiable and convex continuous, respectively, and the sets of *admissible strategies* A_i, $i = 1, 2, \ldots, l$, are closed and convex. The objective in (2) is thus the sum of a smooth function depending on the whole strategy profile $x := (x_1, x_2, \ldots, x_l)$ and a convex (not necessarily smooth) function depending only on x_i.

Let us recall that, given a parameter vector $\bar{p}$, the strategy profile $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_l)$ is a corresponding *Nash equilibrium* provided

$$\bar{x}_i \in \underset{x_i \in A_i}{\operatorname{argmin}} \left[f_i(\bar{p}, x_i, \bar{x}_{-i}) + q_i(x_i) \right] \quad \text{for all } i.$$

Denote by $S : \mathbb{R}^m \rightrightarrows (\mathbb{R}^n)^l$ the solution mapping which assigns each p the corresponding (possibly empty) set of Nash equilibria. The famous Nash Theorem (Aubin 1998, Theorem 12.2) yields the next statement.

Theorem 1 *Given $\bar{p} \in \mathbb{R}^n$, assume that*

(A1) for all admissible values of x_{-i} functions $f_i(\bar{p}, \cdot, x_{-i})$, $i = 1, 2, \ldots, l$, are convex, and

(A2) sets A_i, $i = 1, 2, \ldots, l$, are bounded.

Then $S(\bar{p}) \neq \emptyset$.

Suppose from now on that (A1) holds true for all p from an open set $\mathcal{B} \subset \mathbb{R}^m$. Then one has that $\mathcal{B} \subset \operatorname{dom} S$ and for $p \in \mathcal{B}$

$$S(p) = \{x \,|\, 0 \in F(p, x) + Q(x)\}, \tag{3}$$

where

$$F(p, x) = \begin{bmatrix} F_1(p, x) \\ \vdots \\ F_l(p, x) \end{bmatrix} \text{ with } F_i(p, x) = \nabla_{x_i} f_i(p, x_i, x_{-i}),\ i = 1, 2, \ldots, l, \quad \text{and}$$

$$Q(x) = \partial \tilde{q}(x) \text{ with } \tilde{q}(x) = \sum_{i=1}^{l} \tilde{q}_i(x_i) \text{ and } \tilde{q}_i(x_i) = q_i(x_i) + \delta_{A_i}(x_i),\ i = 1, 2, \ldots, l.$$

This follows immediately from the fact that under the posed assumptions the solution set of (2) is characterized by the first-order condition

$$0 \in \nabla_{x_i} f_i(p, x_i, x_{-i}) + \partial \tilde{q}_i(x_i),\ i = 1, 2, \ldots, l.$$

Remark 1 The GE in (3) can be equivalently written down in the form:
For a given $\bar{p}$ find $\bar{x}$ such that

$$\langle F(\bar{p}, \bar{x}), x - \bar{x} \rangle + \tilde{q}(x) - \tilde{q}(\bar{x}) \geq 0 \quad \text{for all } x.$$

Our equilibrium is thus governed by a variational inequality (VI) of the second kind, cf. (Kinderlehrer and Stampacchia 1980; Facchinei and Pang 2003, page 96).

Next we will concentrate on the (local) analysis of S (given by (3)) under the less restrictive assumptions that, with $s := l\,n$,

(i) $F : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^s$ is continuously differentiable, and
(ii) $Q(\cdot) = \partial \tilde{q}(\cdot)$ for a proper convex, lsc function $\tilde{q} : \mathbb{R}^s \to \overline{\mathbb{R}}$.

In this way, the obtained results will be applicable not only to the equilibrium problem stated above, but to a broader class of parameterized VIs of the second kind. Note that Lipschitzian stability of the *generalized equation* (GE)

$$0 \in F(p, x) + \partial \tilde{q}(x) \tag{4}$$

has been investigated, among other works, in (Rockafellar and Wets 1998, Chapter 13) even without any convexity assumptions imposed on $\tilde{q}$. As proved in (Rockafellar and Wets 1998, Theorem 13.48), S has the Aubin property around $(\bar{p}, \bar{x}) \in \operatorname{gph} S$ provided the *adjoint* GE

$$0 \in \nabla_x F(\bar{p}, \bar{x})^T u + \partial^2 \tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(u) \tag{5}$$

in variable $u \in \mathbb{R}^s$ has only the trivial solution $u = 0$.

This condition is automatically fulfilled provided $\nabla_x F(\bar{p}, \bar{x})$ is positive definite. Indeed, when we premultiply the adjoint GE (5) by u, one obtains that

$$0 = \langle \nabla_x F(\bar{p}, \bar{x})u, u\rangle + \langle u, v\rangle \quad \text{for some } v \in \partial^2 \tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(u).$$

Due to the assumptions imposed on $\tilde{q}$, mapping $\partial \tilde{q}$ is maximal monotone (Rockafellar and Wets 1998, Theorem 12.17). We can thus invoke (Poliquin and Rockafellar 1998, Theorem 2.1), according to which

$$\langle u, v\rangle \geq 0 \ \text{ for all } \ v \in \partial^2 \tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(u).$$

The result thus follows from the positive definiteness of $\nabla_x F(\bar{p}, \bar{x})$.

Let us now derive conditions ensuring the existence of a single-valued and Lipschitzian localization of S around $(\bar{p}, \bar{x})$. To this purpose we employ (Dontchev and Rockafellar 2014, Theorem 3G.4), according to which this property of S is implied by the existence of a single-valued and Lipschitzian localization of the associated partially linearized mapping $\Sigma : \mathbb{R}^s \rightrightarrows \mathbb{R}^s$ defined by

$$\Sigma(w) := \{x | w \in F(\bar{p}, \bar{x}) + \nabla_x F(\bar{p}, \bar{x})(x - \bar{x}) + \partial \tilde{q}(x)\} \tag{6}$$

around $(0, \bar{x})$. This implication leads immediately to the next statement.

Proposition 1 *Assume that $\nabla_x F(\bar{p}, \bar{x})$ is positive definite. Then S has a single-valued and Lipschitzian localization around $(\bar{p}, \bar{x})$.*

Proof Observe first that, by (Rockafellar and Wets 1998, Theorem 13.48), Σ has the Aubin property around $(0, \bar{x})$ if and only if (5) has only the trivial solution $u = 0$ which, in turn, is ensured by the positive definiteness of $\nabla_x F(\bar{p}, \bar{x})$. So the assertion follows from (Dontchev and Rockafellar 2014, Theorem 3G.5) provided the mapping

$$\Phi(x) := F(\bar{p}, \bar{x}) + \nabla_x F(\bar{p}, \bar{x})(x - \bar{x}) + \partial \tilde{q}(x)$$

is *locally monotone* at $(\bar{x}, 0)$, i.e., for some neighborhood $\mathcal{U}$ of $(\bar{x}, -F(\bar{p}, \bar{x}))$, one has

$$\langle x' - x, \nabla_x F(\bar{p}, \bar{x})(x' - x)\rangle + \langle x' - x, y' - y\rangle \geq 0 \;\forall\, (x, y), (x', y') \in \operatorname{gph} \partial\tilde{q} \cap \mathcal{U}.$$

This holds trivially due to the posed assumptions and we are done. □

Under the positive semidefiniteness of $\nabla_x F(\bar{p}, \bar{x})$ one can, moreover, derive a formula for the graphical derivative of S. Indeed, denoting

$$M(p, x) := F(p, x) + \partial\tilde{q}(x),$$

it is easy to see that condition (4.8) of (Gfrerer and Outrata 2016, Corollary 4.5) is fulfilled. It follows that for all $h \in \mathbb{R}^m$

$$\begin{aligned} DS(\bar{p}, \bar{x})(h) &= \{k \in \mathbb{R}^s | 0 \in DM(\bar{p}, \bar{x}, 0)(h, k)\} \\ &= \{k \in \mathbb{R}^s | 0 \in \nabla_p F(\bar{p}, \bar{x})h + \nabla_x F(\bar{p}, \bar{x})k + D\partial\tilde{q}(x, -F(\bar{p}, \bar{x}))(k)\}, \end{aligned}$$

where we have employed the sum rule stated in (Dontchev and Rockafellar 2014, Proposition 4A.2). It remains to make use of (Rockafellar and Wets 1998, Theorem 13.40), thanks to which, under our assumptions, one has for all $h \in \mathbb{R}^m$ the formula

$$DS(\bar{p}, \bar{x})(h) = \{k \in \mathbb{R}^s | 0 \in \nabla_p F(\bar{p}, \bar{x})h + \nabla_x F(\bar{p}, \bar{x})k + \partial\varphi(k)\}, \tag{7}$$

where

$$\varphi(k) := \frac{1}{2} d^2 q(\bar{x} | -F(\bar{p}, \bar{x}))(k).$$

Formula (7) is illustrated in the Apendix by a simple example. The graphical derivative $DS(\bar{p}, \bar{x})$ can be useful in local analysis of S, e.g., via a continuation method, cf. (Allgower and Georg 1997).

In some situations the assumption of positive definiteness of $\nabla_x F(\bar{p}, \bar{x})$ can be weakened. The respective statements are presented also in the Appendix.

4 Optimal strategies of producers

As stated in the Introduction, our motivation for a study of mapping (3) came from an attempt to optimize the production strategies of firms with respect to changing external parameters like input prices, parameters of inverse demand functions etc. These parameters evolve in time and the corresponding adjustments of production strategies have (at least at some producers) to take into account the already mentioned costs of change. The appropriate variant of the GE in (3) (depending on the considered type of market) has thus to be solved at each time step with the updated values of the parameters. In this section we will analyze from this point of view a standard oligopolistic market described thoroughly in Murphy et al. (1982) and in (Outrata et al. 1998, Chapter 12). So, in the framework (2) we will assume that n is the number of produced homogeneous commodities, $p = (p_1, p_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, $m_1 + m_2 = m$ and

$$f_i(p, x_i, x_{-i}) = c_i(p_1, x_i) - \langle x_i, \pi(p_2, T)\rangle \tag{8}$$

with $T = \sum_{i=1}^{l} x_i$. Functions $c_i : \mathbb{R}^{m_1} \times \mathbb{R}^n \to \mathbb{R}$ represent the *production costs* of the ith producer and $\pi : \mathbb{R}^{m_2} \times \mathbb{R}^n \to \mathbb{R}$ is the *inverse demand function* which assigns each value of the parameter p_2 and the overall production vector T the price at which the (price-taking) consumers are willing to demand. Additionally, we assume that, with some non-negative reals β_i,

$$q_i(x_i) = \beta_i \|x_i - a_i\|, \quad i = 1, 2, \ldots, l, \tag{9}$$

where $||\cdot||$ stands for an arbitrary norm in $\mathbb{R}^n$. Sets $A_i \subset \mathbb{R}^n$ specify the *sets of feasible productions* and functions q_i represent the costs of change associated with the change of production from a given vector a_i to x_i. Thus

$$a_i \in A_i, \quad i = 1, 2, \ldots, l,$$

are "previous" productions which have to be changed taking into account the "new" values of parameters p_1, p_2. Clearly, one could definitely work also with more complicated functions q_i. Let us denote the total costs (negative profits) of the single firms by

$$J_i(p, x_i, x_{-i}) := f_i(p, x_i, x_{-i}) + q_i(x_i), \quad i = 1, 2, \ldots, l.$$

In accordance with Murphy et al. (1982) and Outrata et al. (1998) we will now assume for brevity that $n = 1$ (so that $s = l$) and impose the following assumptions:

(S1) $\exists$ an open set $\mathcal{B}_1 \subset \mathbb{R}^{m_1}$ and open sets $\mathcal{D}_i \supset A_i$ such that for for $i = 1, 2, \ldots, l$

- c_i are twice continuously differentiable on $\mathcal{B}_1 \times \mathcal{D}_i$;
- $c_i(p_1, \cdot)$ are convex for all $p_1 \in \mathcal{B}_1$.

(S2) $\exists$ an open set $\mathcal{B}_2 \subset \mathbb{R}^{m_2}$ such that

- π is twice continuously differentiable on $\mathcal{B}_2 \times \text{int}\mathbb{R}_+$ and $\pi(p_2, \cdot)$ is strictly convex on $\text{int}\mathbb{R}_+$ for all $p_2 \in \mathcal{B}_2$;
- $\vartheta\pi(p_2, \vartheta)$ is a concave function of ϑ for all $p_2 \in \mathcal{B}_2$.

(S3) Sets $A_i \subset \mathbb{R}_+$ are closed bounded intervals and at least one of them belongs to $\text{int}\mathbb{R}_+$.

Note that thanks to (S3) one has that $T > 0$ for any feasible production profile

$$(x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$$

and hence the second term in (8) (representing the revenue) is well-defined.

By virtue of (Outrata et al. 1998, Lemmas 12.1 and 12.2) we conclude that, with f_i and q_i given by (8) and (9), respectively, and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, the assumptions of Proposition 1 are fulfilled. This means in particular that for all vectors $(a_1, a_2, \ldots, a_l) \in A_1 \times A_2 \times \ldots \times A_l$ the respective mapping $S : (p_1, p_2) \mapsto x$ has a single-valued and Lipschitzian localization around any triple (p_1, p_2, x), where $(p_1, p_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ and $x \in S(p_1, p_2)$. Under the posed assumptions, however, a stronger statement can be established.

Theorem 2 *Let $a \in A_1 \times A_2 \times \ldots \times A_l$. Under the posed assumptions (S1)-(S3) the solution mapping S is single-valued and locally Lipschitzian over $\mathcal{B}_1 \times \mathcal{B}_2$.*

Proof Given the vectors a_i, $i = 1, 2, \ldots, l$, and the parameters p_1, p_2, the GE in (3) attains the form

$$\begin{aligned} 0 \in & \begin{bmatrix} \nabla_{x_1} c_1(p_1, x_1) - x_1 \nabla_{x_1} \pi(p_2, T) - \pi(p_2, T) \\ \vdots \\ \nabla_{x_l} c_l(p_1, x_l) - x_l \nabla_{x_l} \pi(p_2, T) - \pi(p_2, T) \end{bmatrix} \\ & + \begin{bmatrix} \Lambda_1(x_1 - a_1) \\ \vdots \\ \Lambda_l(x_l - a_l) \end{bmatrix} + N_{A_1}(x_1) \times \ldots \times N_{A_l}(x_l), \end{aligned} \tag{10}$$

where

$$\Lambda_i(x_i - a_i) = \begin{cases} \beta_i & \text{if } x_i > a_i \\ -\beta_i & \text{if } x_i < a_i \\ [-\beta_i, \beta_i] & \text{otherwise.} \end{cases}$$

From (Outrata et al. 1998, Lemma 12.2) and (Rockafellar and Wets 1998, Proposition 12.3) it follows that for any $(p_1, p_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ the first mapping on the right-hand side of (10) is strictly monotone in variable x. Moreover, the second one, as the subdifferential of a proper convex function is monotone (Rockafellar and Wets 1998, Theorem 12.17). Their sum is strictly monotone by virtue of (Rockafellar and Wets 1998, Exercise 12.4(c)) and so we may recall (Rockafellar and Wets 1998, Example 12.48) according to which $S(p_1, p_2)$ can have no more than one element for any $(p_1, p_2) \in \mathcal{B}_1 \times \mathcal{B}_2$. This, combined with Theorem 1 and the Lipschitzian stability of S mentioned above proves the result. □

In fact, under the posed assumptions the changes of the (unique) Cournot-Nash equilibrium are proportional to (small) perturbations of *all* data which enter the single-valued part of GE (10) in the continuously differentiable way. This is a favourable situation for a possible application of post-optimal analysis when one estimates this dependence on possibly uncertain data also quantitatively.

In the next section we will be dealing with the mapping $Z_{a,p_1,p_2} : \mathbb{R} \rightrightarrows \mathbb{R}^{l-1}$ which, for given fixed values of a, p_1 and p_2, assigns each vector $x_1 \in A_1$ a solution

$(x_2, \ldots, x_l)$ of the GE

$$0 \in \begin{bmatrix} \nabla_{x_2} c_2(p_1, x_2) - \langle x_2 \nabla_{x_2} \pi(p_2, T)\rangle - \pi(p_2, T) \\ \vdots \\ \nabla_{x_l} c_l(p_1, x_l) - \langle x_l \nabla_{x_l} \pi(p_2, T)\rangle - \pi(p_2, T) \end{bmatrix} + \begin{bmatrix} \Lambda_2(x_2 - a_2) \\ \vdots \\ \Lambda_l(x_l - a_l) \end{bmatrix} + N_{A_2}(x_2) \times \ldots \times N_{A_l}(x_l). \tag{11}$$

Variable x_1 enters GE (11) via $T (= \sum_{i=1}^{l} x_i)$. Using the same argumentation as in Theorem 2 we obtain the following result.

Theorem 3 *Let $a_i \in A_i$ for $i = 2, 3, \ldots, l$, $p_1 \in \mathcal{B}_1$ and $p_2 \in \mathcal{B}_2$ be given. Then, under the assumptions of Theorem 2, mapping Z_{a,p_1,p_2} is single-valued and locally Lipschitzian over A_1.*

This statement enables us to consider the situation when the first producer decides to replace the non-cooperative by the Stackelberg strategy, cf. (Outrata et al. 1998, page 220). In this case, to maximize his profit, he has, for the given values of a, p_1 and p_2, to solve a bilevel game which, in the considered case, amounts to the *mathematical program with equilibrium constraint* (MPEC)

$$\begin{aligned} &\underset{x_1}{\text{minimize}} \quad c_1(p_1, x_1) - x_1 \pi(p_2, T) + q_1(x_1) \\ &\text{subject to} \\ &\qquad\qquad x_{-1} \text{ is a solution of (11)} \\ &\qquad\qquad x_1 \in A_1. \end{aligned} \tag{12}$$

Thanks to Theorem 3 problem (12) can be replaced by the (single-level) nonsmooth minimization problem

$$\begin{aligned} &\underset{x_1}{\text{minimize}} \quad \Theta_{a,p_1,p_2}(x_1) \\ &\text{subject to} \\ &\qquad\qquad x_1 \in A_1. \end{aligned} \tag{13}$$

In (13), $\Theta_{a,p_1,p_2} : \mathbb{R} \to \mathbb{R}$ is the composition defined by

$$\Theta_{a,p_1,p_2}(x_1) = c_1(p_1, x_1) - x_1 \pi(p_2, x_1 + \mathcal{L}(Z(x_1))) + q_1(x_1), \tag{14}$$

where the mapping $\mathcal{L} : \mathbb{R}^{l-1} \to \mathbb{R}$ is defined by

$$\mathcal{L}(x_2, x_3, \ldots, x_l) = \sum_{i=2}^{l} x_i.$$

and a, p_1, p_2 amount to fixed parameters. Problem (13) is thus a minimization of a locally Lipschitzian function to which various numerical approaches can be applied.

5 Numerical methods and results

There are several approaches applicable to the numerical solution of problems (10),(11) for fixed values of a, p_1, p_2 and a, p_1, p_2, x_1, respectively. Thanks to the posed assumptions, one can choose, for instance, a suitable method from (Facchinei and Pang 2003, Chapter 12). Alternatively, the Gauss or Gauss-Seidel (GS) methods described in Kanzow and Schwartz (2018), coupled with a nonsmooth optimization routine, could be adapted to this aim and also the general Newton-type method from Gfrerer and Outrata (2019) can be specialized to (even more general) variational inequalities of the second kind. In this section we will first describe an implementation of the forward-backward splitting (FB) method from (Facchinei and Pang 2003, page 1153) which, due to the separable structure of q, requires essentially only a repetitive solution of very simple univariate convex optimization problems. Thereafter, in Section 5.1, we will use this method to compute the Cournot-Nash equilibria for data, taken over from Murphy et al. (1982); Outrata et al. (1998). Finally, in Section 5.2, we will compute the corresponding "Stackelberg-Cournot-Nash" equilibria via the *implicit programming approach* (ImP), where the FB method will be used inside a simple nonsmooth optimization routine.

Consider now GE (10), where we denote, to unburden the notation, the single-valued part by $F_{p_1,p_2}(x)$ and the multi-valued part by $Q_a(x)$. Clearly, the ith component of Q_a amounts to

$$Q_a^i = \Lambda_i(x_i - a_i) + N_{A_i}(x_i), \quad i = 1, 2, \ldots, l.$$

It is easy to see that, given some $c > 0$, the resolvent $J_{c\,Q_a}$ of Q_a (with constant c) at an argument z has the value

$$J_{c\,Q_a}(z) = y,$$

where the component y_i, $i = 1, 2, \ldots, l$, of y is the (unique) solution of the univariate optimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} y_i^2 - z_i y_i + c\,\beta_i |y_i - a_i| \\ \text{subject to} & \\ & y_i \in A_i . \end{array} \tag{15}$$

Since problems (15) can easily be solved exactly, we can employ the following (exact) variant of the FB method, where for the sake of notational simplicity the indices at F and Q are omitted.

Algorithm 1 (FB method)

```
1: initialization: ε > 0, c > 0, k = 0, x^0 = (x_1^0, x_2^0, ..., x_l^0) ∈ A_1 × A_2 × ··· × A_l
2: if dist(−F^i(x_i^k), Q^i(x_i^k)) ≤ ε for all i = 1, 2, , ..., l, then
3:     stop
4: end if
5: compute z^k = x^k − cF(x^k)
6: for i=1,2,..., l do
7:     solve problem (15) arriving at the solution y_i
8: end for
9: set x^{k+1} = y, k = k + 1 and go to 2
```

By virtue of (Facchinei and Pang 2003, Theorem 12.4.6) the imposed assumptions ensure that Algorithm 1 is well defined and generates a sequence $\{x^k\}$ converging to the unique Cournot-Nash equilibrium. Indeed, F_{p_1,p_2} is not only strictly monotone (as pointed out in the proof of Theorem 2), but even strongly monotone in variable x on the bounded set $A_1 \times A_2 \times \cdots \times A_l$. This follows from (Outrata et al. 1998, Lemma 12.2) taking into account that the mapping, which assigns x the lowest eigenvalue of the symmetrized Jacobian of F_{p_1,p_2}, is continuous and applying the compactness argument. This strong monotonicity implies in particular that F_{p_1,p_2} is co-coercive on dom Q_a(Zhu and Marcotte 1996, Definition 1). Note that the admissible choice of parameter c is related to the respective modulus of co-coercivity. By (Facchinei and Pang 2003, Corollary 12.4.8) the sequence $\{x^k\}$ converges in fact at least R-linearly.

5.1 Cournot-Nash equilibria

We consider an example from (Outrata et al. 1998, Section 12.1) enhanced by a nonsmooth term reflecting the cost of change. We have five firms (i.e., $l = 5$) supplying production quantities (productions)

$$(x_1, x_2, \ldots, x_5) \in A_1 \times A_2 \times \cdots \times A_5$$

of one (i.e., $n = 1$) homogeneous commodity to a common market and

$$A_1 = [1, 150], \quad A_2 = \cdots = A_5 = [0, 150].$$

are production bounds. Further we assume a market characterized by the inverse demand function

$$\pi(\gamma, T) = 5000^{1/\gamma}\, T^{-1/\gamma},$$

where γ is a positive parameter termed *demand elasticity* and

$$T = x_1 + x_2 + \cdots + x_5$$

denotes the total production of all firms. In our tests, however, this parameter will be fixed ($\gamma = 1$). The resulting inverse demand function is depicted in Fig. 1 (left).

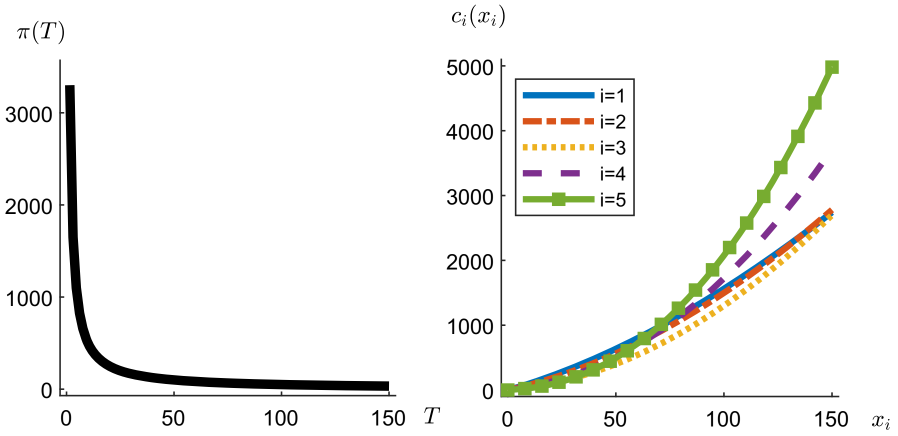

Fig. 1 The inverse demand function $\pi(\gamma, T)$ with $\gamma = 1$ (left) and production cost functions $c_i(x_i)$, $i = 1, \dots, 5$, for $t = 1$ (right)

The production cost functions have the form

$$c_i(b_i, x_i) = b_i x_i + \frac{\delta_i}{\delta_i + 1} K_i^{-1/\delta_i} x_i^{(1+\delta_i)/\delta_i},$$

where $b_i, \delta_i, K_i, i = 1, 2, \dots, 5$, are positive parameters. For brevity we assume that only the parameters b_i, reflecting the impact of the input prices on the production costs, evolve in time, whereas parameters δ_i, K_i attain the same constant values as in (Outrata et al. 1998, Table 12.1). The cost of change

$$q_i(x_i) = \beta_i |x_i - a_i|, \quad i = 1, 2, \dots, 5,$$

will arise only at firms 1, 2 and 3 with different multiplicative constants

$$\beta_1 = 0.5, \quad \beta_2 = 1, \quad \beta_3 = 2.$$

At the remaining firms any change of production does not incur additional costs ($\beta_4 = \beta_5 = 0$). We will study the behaviour of the market over three time intervals, $t \in \{1, 2, 3\}$ with the initial productions (at $t = 0$)

$$a_1 = 47.81, \quad a_2 = 51.14, \quad a_3 = 51.32, \quad a_4 = 48.55, \quad a_5 = 43.48,$$

corresponding to the standard Cournot-Nash equilibrium with the parameters taken over from Murphy et al. (1982). The production cost functions for $t = 1$ are depicted in Fig. 1 (right) and the evolution of parameters b_i is displayed in Table 1.

For the computation of the Cournot-Nash equilibria in the single time instances we used Algorithm 1 with constant $c = 2$ and the initial iterate $x^0 = (75, 75, \dots, 75)$ in the first time step. In the subsequent time steps we used, as the initial iterate, the

Table 1 Time dependent input parameters b_i for the production costs

	i	1	2	3	4	5
t=1	b_i	9	7	3	4	2
t=2	b_i	10	8	5	4	2
t=3	b_i	11	9	8	4	2

Table 2 Cournot-Nash equilibria x_i, the corresponding negative total costs $-J_i$ (profits) and costs of change $[q_i]$

	i	1	2	3	4	5
t=0	a_i	47.81	51.14	51.32	48.55	43.48
t=1	x_i	49.41	51.14	54.24	48.05	43.09
	$-J_i$ $[q_i]$	377.23 [+0.80]	459.95	639.95 [+5.83]	503.44	507.09
t=2	x_i	49.41	51.14	54.24	48.05	43.09
	$-J_i$ $[q_i]$	328.62	408.81	537.30	503.44	507.09
t=3	x_i	45.71	51.14	51.58	48.76	43.64
	$-J_i$ $[q_i]$	286.75 [+1.85]	379.76	386.92 [+5.31]	527.22	527.81

Cournot-Nash equilibrium from the preceding one (which is justified due to Theorem 2).

We observe the following number of iterations of Algorithm 1 (corresponding to times $t \in \{1, 2, 3\}$) with respect to the required accuracy ε:

- $\{26, 0, 23\}$ iterations for $\varepsilon = 10^{-6}$,
- $\{34, 0, 31\}$ iterations for $\varepsilon = 10^{-8}$,
- $\{42, 0, 38\}$ iterations for $\varepsilon = 10^{-10}$.
- $\{50, 0, 46\}$ iterations for $\varepsilon = 10^{-12}$.
- $\{59, 0, 54\}$ iterations for $\varepsilon = 10^{-14}$.

The obtained results are summarized in Table 2 and show the productions and profits (negative total costs) of all firms at single time instances. In parentheses we display the costs of change which decrease the profits of the firms 1, 2, 3 in case of any change of their production strategies.

Note that firms 1 and 3 increased significantly their productions in time 1 and decreased them in time 3, whereas firm 2, due to the cost of change, kept its production unchanged during the whole time.

Figure 2 shows the total cost functions J_i at time 1. Note that the equilibrium production of firm 2 lies in a kink point because its cost of change is zero. Expectantly, functions J_i are smooth for $i = 4, 5$.

5.2 Stackelberg-Cournot-Nash equilibria

Next we will consider the same market as in the previous section where, however, the first producer decides now to replace the non-cooperative by the Stackelberg strategy. The application of the ImP approach leads to problem (13) which is, however, more

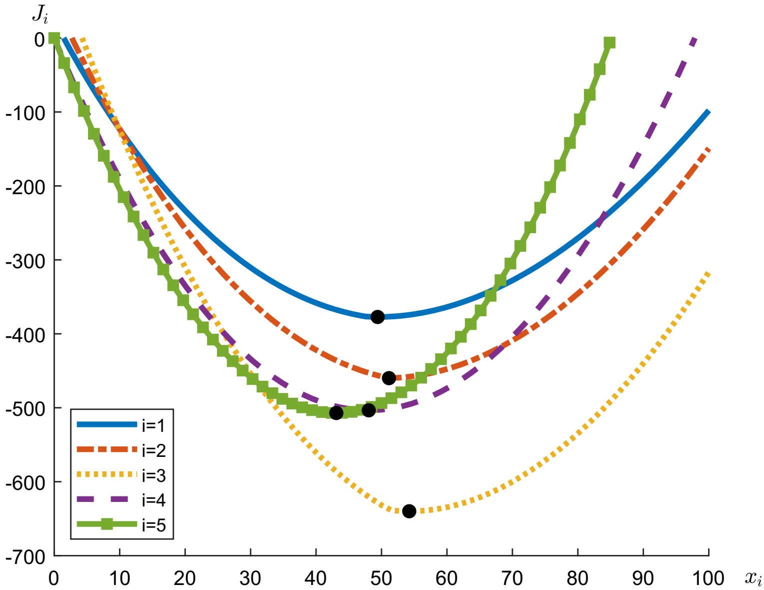

Fig. 2 The total cost functions J_i, $i = 1, \ldots, 5$ and equilibrium productions indicated by bullets for $t = 1$

complicated than its counterparts solved via this approach in (Outrata et al. 1998, Chapter 12). Indeed, the presence of costs of change makes the computation of the lower-level equilibrium more difficult and the objective of the Leader is, in contrast to Outrata et al. (1998), not continuously differentiable. Generally, the ImP approach is not applicable provided the mapping Z_{a,p_2,p_2} is not single-valued and locally Lipschitzian. In such a case, different approaches are available, see, e.g., Basilico et al. (2020). Many further useful references can be found in Dempe (2018).

On the other hand, since $n = 1$, we may use for the minimization of Θ_{a,p_1,p_2} any suitable routine for nonsmooth constrained univariate minimization. In our computations we used the inbuilt Matlab function `fminbnd` (Brent 1973).

The obtained results are summarized in Table 3. They are quite different from their counterparts in Table 2 and show that, switching to the Stackelberg strategy, firm 1 substantially improves its profit. In contrast to the noncooperative strategy, it has now to change its production at each time step and also firm 2, who preserved in the Cournot-Nash case the same production over the whole time, is now forced to change it at $t = 3$. Of course, our data are purely academic and can hardly be used for some economic interpretations. On the other hand, the results are sound and show the potential of the suggested techniques in applications to some more realistic situations.

Remark 2 (Gauss-Seidel method) Alternatively, instead of using the FB method described in Algorithm 1, we applied both in Sections 5.1 and 5.2 a “nonsmooth” variant of the Gauss-Seidel (GS) method from Kanzow and Schwartz (2018) using the same initial iterate and the same ε in the stopping criterion. The univariate problems solved in the framework of the GS method are more complicated than problems (15) and, to solve them, we again used the inbuilt Matlab function `fminbnd`. We obtained

Table 3 Stackelberg-Cournot-Nash equilibria x_i, the corresponding negative total costs $-J_i$ (profits) and costs of change $[q_i]$. Firm 1 is a Leader

	i	1	2	3	4	5
t=0	a_i	47.81	51.14	51.32	48.55	43.48
t=1	x_i	54.95	51.14	53.59	47.52	42.68
	$-J_i$ $[q_i]$	380.49 [+3.57]	443.52	619.80 [+4.54]	486.00	491.88
t=2	x_i	53.09	51.14	53.59	47.72	42.84
	$-J_i$ $[q_i]$	329.49 [+0.93]	398.58	523.65	492.55	497.60
t=3	x_i	53.05	50.46	50.77	48.11	43.14
	$-J_i$ $[q_i]$	289.65 [+0.02]	356.57 [+0.68]	364.33 [+5.64]	505.29	508.71

essentially the same results as those displayed above in Table 2. We observe the following number of iterations of the GS method (corresponding to times $t \in \{1, 2, 3\}$) with respect to the required accuracy ε- $\{7, 0, 6\}$ iterations for $\varepsilon = 10^{-6}$, but the required accuracy was not achieved for smaller values (such as $\varepsilon = 10^{-8}$) . In our opinion, the FB method is favourable, because the convergence of the GS method may be problematic, cf. (Kanzow and Schwartz 2018, Sections 6.3, 6.4).

Remark 3 All numerical results were generated by own Matlab code available for download at: https://www.mathworks.com/matlabcentral/fileexchange/72771 . The code is flexible and allows for easy modifications to different models.

Conclusion

In the first half of the paper we have studied a parametrized variational inequality of the second kind. In this form, one can write down, for example, a condition which characterizes solutions of some parameter-dependent Nash equilibrium problems. By using standard tools of variational analysis sufficient conditions have been derived ensuring the existence of a single-valued and Lipschitzian localization of the respective solution mappings. These conditions can be useful in post-optimal analysis and we used them in Section 5.2 when computing the Stackelberg-Cournot-Nash equilibria by the implicit programming approach. Some further stability results concerning GE (4) are presented in the Appendix below.

The second part of the paper has been inspired, on the one hand, by the successful theory of rate-independence processes (Mielke and Roubíček 2015; Frost et al. 2019) and, on the other hand, by the important economic paper Flåm (2020). It turns out that in some market models the cost of change of the production strategy can be viewed as the economic counterpart of the dissipation energy, arising in rate independent dissipative models of nonlinear mechanics of solids. Cost of change (dissipation energy) occurs further, e.g., in modeling the behavior of some national banks who try to regulate the inflation rate, among other instruments, via buying or selling suitable amounts of the domestic currency on international financial markets.

The FB method which we use in Sections 5.1, 5.2 for the computation of Cournot-Nash and Stackelberg-Cournot-Nash equilibria, respectively, seems to fit well the structure of GE(10) and GE(11). Due to the separability of q, namely, the computation of the appropriate resolvent reduces to l exact formulas, which makes the computational effort per iteration very low. Of course, if the number of produced commodities increases, we will have to solve problems (15) via a suitable nonsmooth minimization technique.

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APPENDIX

In some models of practical importance function q is piecewise linear-quadratic. Then the assumption of positive definiteness of $\nabla_x F(\bar{p}, \bar{x})$ in Proposition 1 can be somewhat relaxed.

Proposition 2 *Assume that $\tilde{q}$ is convex, piecewise linear-quadratic and the mapping $\Xi : \mathbb{R}^s \rightrightarrows \mathbb{R}^s$ defined by*

$$\Xi(w) := \left\{k \in \mathbb{R}^s | w \in \nabla_x F(\bar{p}, \bar{x})k + \partial\varphi(k)\right\} \tag{16}$$

with $\varphi(k) := \frac{1}{2} d^2 q(\bar{x}| - F(\bar{p}, \bar{x}))(k)$ is single-valued on $\mathbb{R}^s$. Then S has a single-valued and Lipschitzian localization around $(\bar{p}, \bar{x})$.

Proof By virtue of (Dontchev and Rockafellar 2014, Theorem 3G.4) it suffices to show that the single-valuedness of Ξ implies the existence of a single-valued and Lipschitzian localization of Σ (defined in (6)) around $(0, \bar{x})$. Clearly,

$$\text{gph}\, \Sigma = \left\{(w, x) \middle| \begin{bmatrix} x - \bar{x} \\ w - \nabla_x F(\bar{p}, \bar{x})(x - \bar{x}) \end{bmatrix} \in \text{gph}\, \partial\tilde{q} - \begin{bmatrix} \bar{x} \\ -F(\bar{p}, \bar{x}) \end{bmatrix}\right\}$$

so that Σ is a polyhedral multifunction due to our assumptions imposed on $\tilde{q}$, cf. (Rockafellar and Wets 1998, Theorem 12.30). It follows from Robinson (1976) (see also (Outrata et al. 1998, Cor.2.5)) that due to the polyhedrality of Σ, it suffices to ensure the single-valuedness of $\Sigma(\cdot) \cap \mathcal{V}$ on $\mathcal{U}$, where $\mathcal{U}$ is a convex neighborhood of $0 \in \mathbb{R}^s$ and $\mathcal{V}$ is a neighborhood of $\bar{x}$. Let us select these neighborhoods in such a way that

$$\text{gph}\, \partial\tilde{q} - \begin{bmatrix} \bar{x} \\ -F(\bar{p}, \bar{x}) \end{bmatrix} = T_{\text{gph}\, \partial\tilde{q}}(\bar{x}, -F(\bar{p}, \bar{x})),$$

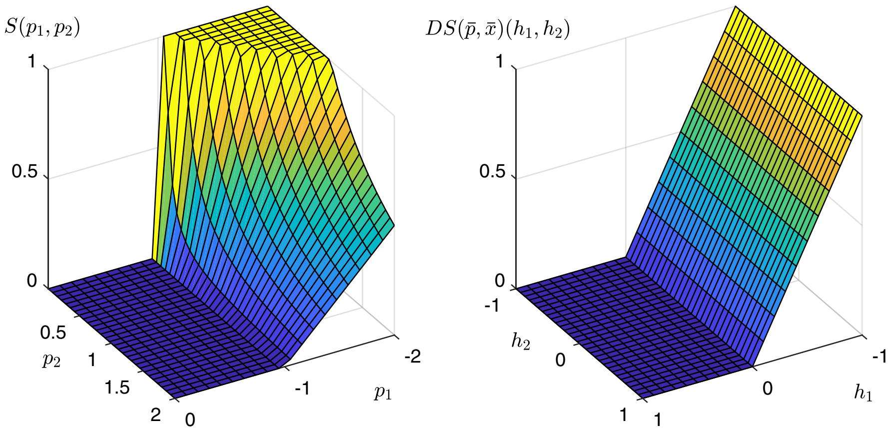

Fig. 3 The set in equilibria $S(p_1, p_2)$ (left) and the graphical derivative $DS(\bar{p}, \bar{x})(h_1, h_2)$ of Example 1

which is possible due to the polyhedrality of $\partial\tilde{q}$. Then one has

$$\text{gph}\,\Sigma \cap (\mathcal{U} \times \mathcal{V}) = \{(w, \bar{x} + k) \in \mathcal{U} \\ \times \mathcal{V} | w \in \nabla_x F(\bar{p}, \bar{x})k + D\partial\tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(k)\}.$$

Under the posed assumptions for any $k \in \mathbb{R}^n$

$$D\partial\tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(k) = \partial\varphi(k),$$

cf. (Rockafellar and Wets 1998, Theorem 13.40), so that gph $\Sigma \cap (\mathcal{U} \times \mathcal{V}) = \{(w, \bar{x} + k) \in \mathcal{U} \times \mathcal{V} | (w, k) \in \text{gph}\,\Xi\}$. Since $D\partial\tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(\cdot)$ is positively homogeneous, $\partial\varphi(\cdot)$ is positively homogeneous as well and so the single-valuedness of $\Sigma(\cdot) \cap \mathcal{V}$ on $\mathcal{U}$ amounts exactly to the single-valuedness of Ξ on $\mathbb{R}^s$. □

On the basis of (Rockafellar and Wets 1998, Proposition 13.9) the single-valuedness of Ξ can be ensured via the notion of copositivity. Recall that an $[s \times s]$ matrix H is *strictly copositive* with respect to a cone $\mathcal{K} \subset \mathbb{R}^s$ provided

$$\langle d, Hd\rangle > 0 \quad \text{for all } d \in \mathcal{K}, d \neq 0.$$

Proposition 3 *Assume that $\tilde{q}$ is convex, piecewise linear-quadratic and $\tilde{q}''(\bar{x}; \cdot)$ is convex. Further suppose that $\nabla_x F(\bar{p}, \bar{x})$ is strictly copositive with respect to $K - K$, where*

$$K := \{k | \tilde{q}'(\bar{x}; k) = \langle -F(\bar{p}, \bar{x}), k\rangle\}.$$

Then S has a single-valued and Lipschitzian localization around $(\bar{p}, \bar{x})$.

Proof By virtue of (Rockafellar and Wets 1998, Proposition 13.9) the second subderivative $d^2\tilde{q}(\bar{x}|-F(\bar{p},\bar{x}))(\cdot)$ is proper convex and piecewise linear-quadratic and one has

$$\partial\varphi(k) = \partial\frac{1}{2}d^2\tilde{q}(\bar{x}|-F(\bar{p},\bar{x}))(k) = \partial\frac{1}{2}\tilde{q}''(\bar{x};k) + N_K(k). \tag{17}$$

It remains to show that mapping (16) is single-valued. Clearly, the GE in (16) can be written down in the form

$$0 \in \Psi(k) - w + N_K(k),$$

where the multifunction $\Psi(k) := \nabla_x F(\bar{p},\bar{x})k + \partial\frac{1}{2}\tilde{q}''(\bar{x};k)$. As explained in (Outrata et al. 1998, Theorem 4.6), under the posed assumptions there is a positive real α such that

$$\langle d, \nabla_x F(\bar{p},\bar{x})d\rangle \geq \alpha||d||^2 \quad \text{for all } d \in K - K.$$

It follows that for all $k_1, k_2 \in K$, $\xi_1 \in \partial\frac{1}{2}\tilde{q}''(\bar{x};k_1)$, $\xi_2 \in \partial\frac{1}{2}\tilde{q}''(\bar{x};k_2)$ and

$$\eta_1 = \nabla_x F(\bar{p},\bar{x})k_1 + \xi_1 - w, \quad \eta_2 = \nabla_x F(\bar{p},\bar{x})k_2 + \xi_2 - w,$$

one has

$$\begin{aligned}&\langle \eta_1 - \eta_2, k_1 - k_2\rangle \\ &\quad = \langle k_1 - k_2, \nabla_x F(\bar{p},\bar{x})(k_1 - k_2)\rangle + \langle \xi_1 - \xi_2, k_1 - k_2\rangle \geq \alpha||k_1 - k_2||^2.\end{aligned}$$

We conclude that Φ is strongly monotone on K and, consequently, Ξ is single-valued by virtue of (Rockafellar and Wets 1998, Proposition 12.54). □

Example 1 Put $m = 2$, $s = 1$ and consider the GE (4), where

$$F(p,x) = p_1 + p_2 x, \quad \tilde{q}(x) = |x| + \delta_A(x), \quad A = [0,1]$$

and the *reference pair* $(\bar{p},\bar{x}) = ((-1,1),0)$. Since $\nabla_x F(\bar{p},\bar{x}) = 1$, Proposition 1 applies and we may conclude that the respective mapping S has indeed the single-valued and Lipschitzian localization around $(\bar{p},\bar{x})$.

To compute $DS(\bar{p},\bar{x})$, we may employ formula (7), where $\partial\varphi$ is computed according to (17). One has $K(\bar{x},\bar{v}) = \mathbb{R}_+$, $\tilde{q}''(\bar{x},w) = 0$ for any $w \in \mathbb{R}_+$ and so we obtain that

$$\partial\varphi(k) = N_{\mathbb{R}_+}(k).$$

This yields the formula

$$DS(\bar{p},\bar{x})(h) = \{k \in \mathbb{R} | 0 \in h_1 + k + N_{\mathbb{R}_+}(k)\}$$

valid for all $h \in \mathbb{R}^2$. Both mappings S and $DS(\bar{p},\bar{x})$ are depicted in Fig.3. △

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