



Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha



Window-dependent bases for efficient representations of the Stockwell transform

U. Battisti^{a,*}, L. Riba^{a,b}^a *Dipartimento di Matematica, Università degli Studi di Torino, via Carlo Alberto 10, 10123 Torino, Italy*^b *Laboratoire des Fluides Complexes et leurs Réservoirs (LFC-R), Université de Pau et des Pays de l'Adour (UPPA), 64013 Pau, Cedex, France¹*

ARTICLE INFO

Article history:

Received 18 June 2014

Received in revised form 24

December 2014

Accepted 15 February 2015

Available online xxxx

Communicated by Christopher Heil

Keywords:

Stockwell transform

Orthonormal basis

Short-time Fourier transform

Wavelet transform

ABSTRACT

Since its appearing in 1996, the Stockwell transform (S-transform) has been applied to medical imaging, geophysics and signal processing in general. In this paper, we prove that the system of functions (so-called DOST basis) is indeed an orthonormal basis of $L^2([0, 1])$, which is time–frequency localized, in the sense of Donoho–Stark Theorem (1989) [11]. Our approach provides a unified setting in which to study the Stockwell transform (associated with different admissible windows) and its orthogonal decomposition. Finally, we introduce a fast $\mathcal{O}(N \log N)$ – algorithm to compute the Stockwell coefficients for an admissible window. Our algorithm extends the one proposed by Y. Wang and J. Orchard (2009) [33].

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Let f be a signal with finite energy, that is $f \in L^2(\mathbb{R})$, and let φ be a window in $L^2(\mathbb{R})$. Then, following M.W. Wong and H. Zhu [34], we define the Stockwell transform (S-transform) $S_\varphi f$ as

$$(S_\varphi f)(b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) |\xi| \overline{\varphi(\xi(t-b))} dt, \quad b, \xi \in \mathbb{R}. \quad (1.1)$$

It is possible to rewrite the S-transform with respect to the Fourier transform of the analyzed signal:

$$(S_\varphi f)(b, \xi) = \int_{\mathbb{R}} e^{2\pi i b \zeta} \widehat{f}(\zeta + \xi) \overline{\widehat{\varphi}\left(\frac{\zeta}{\xi}\right)} d\zeta, \quad b, \xi \in \mathbb{R}, \quad \xi \neq 0, \quad (1.2)$$

* Corresponding author.

E-mail addresses: ubertino.battisti@unito.it (U. Battisti), luigi.riba@univ-pau.fr (L. Riba).¹ Present affiliation.

where \widehat{f} is the Fourier transform of the signal f , given by

$$\widehat{f}(\xi) = (F f)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) dt, \quad \xi \in \mathbb{R}.$$

We fix the notation: we denote with \check{f} or $F^{-1} f$ the inverse Fourier transform of a signal f . $\mathbb{N} = \{0, 1, \dots\}$ is the set of non-negative integers, $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ is the set of integers.

The S-transform was initially defined by R.G. Stockwell, L. Mansinha and R.P. Lowe in [29] using a Gaussian window

$$g(t) = e^{-t^2/2}, \quad t \in \mathbb{R}.$$

In this case,

$$(S_g f)(b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) |\xi| e^{-(t-b)^2 \xi^2/2} dt, \quad b, \xi \in \mathbb{R}, \quad (1.3)$$

which, in the alternative formulation, becomes

$$(S_g f)(b, \xi) = \int_{\mathbb{R}} e^{2\pi i \zeta b} \widehat{f}(\zeta + \xi) e^{-2\pi^2 \zeta^2/\xi^2} d\zeta, \quad b, \xi \in \mathbb{R}, \quad \xi \neq 0. \quad (1.4)$$

The natural discretization of (1.4), introduced in [29], is given by

$$(S_g f)(j, n) = \sum_{m=0}^{N-1} e^{2\pi i m j/N} \widehat{f}(m+n) e^{-2\pi^2 m^2/n^2}, \quad (1.5)$$

where $j = 0, \dots, N-1$ and $n = 1, \dots, N-1$. For $n = 0$, it is set

$$(S_g f)(j, 0) = \frac{1}{N} \sum_{k=0}^{N-1} f(k), \quad j = 0, \dots, N-1.$$

In the literature, (1.5) is called redundant (discrete) Stockwell transform. Unfortunately, the redundant Stockwell transform has a high computational cost: $\mathcal{O}(N^2 \log N)$. To overcome this problem, R.G. Stockwell introduced in [27], without a mathematical proof, a basis for periodic signals with finite energy, *i.e.* $L^2([0, 1])$, given by

$$\bigcup_{p \in \mathbb{Z}} D_p = \bigcup_{p \in \mathbb{Z}} \{D_{p,\tau}\}_{\tau=0}^{\beta(p)-1}. \quad (1.6)$$

This basis, precisely defined in Section 3, is adapted to octave samples in the frequency domain. The decomposition of a periodic signal f in this basis is called in the literature the discrete orthonormal Stockwell transform (DOST). The related coefficients

$$f_{p,\tau} = (f, D_{p,\tau})_{L^2([0,1])},$$

are called DOST coefficients.

In this paper we prove that this basis is not suited to the standard S-transform with Gaussian window (1.1), rather to an S-transform associated with a characteristic function (boxcar window). This fact was already pointed out by R.G. Stockwell himself in [27] and [28]. The computational complexity of the algorithm suggested by R.G. Stockwell was still high: $\mathcal{O}(N^2)$. In 2009, Y. Wang and J. Orchard [33] proposed a fast algorithm which reduces drastically the complexity to $\mathcal{O}(N \log N)$; the same complexity of the FFT. This achievement allowed a wider application of the S-transform to image analysis.

We provide an adapted basis of $L^2([0, 1])$ on which to decompose the Stockwell transform with a general admissible window φ . Assume that we can find such a basis E_p^φ of $L^2([0, 1])$, depending on the choice of φ . Then, by linearity, we can write

$$(S_\varphi f)(b, \xi) = \sum_p c_p^\varphi (S_\varphi E_p^\varphi)(b, \xi) \tag{1.7}$$

where

$$f = \sum_j c_j^\varphi E_j^\varphi.$$

An ideal basis would satisfy the following properties:

- (i) E_p^φ is an orthonormal basis of $L^2([0, 1])$, so that

$$c_p^\varphi = (f, E_p^\varphi)_{L^2([0,1])};$$

- (ii) $(S_\varphi E_p^\varphi)(b, \xi)$ is *local* in time;
- (iii) $(S_\varphi E_p^\varphi)(b, \xi)$ is *local* in frequency;
- (iv) we can find a fast algorithm ($\mathcal{O}(N \log N)$) to compute the coefficients

$$(f, E_p^\varphi)_{L^2([0,1])}.$$

We prove that (1.6) is indeed an orthonormal basis of $L^2([0, 1])$ satisfying conditions (i), (ii), (iii) and (iv) if $\varphi = \check{\chi} = F^{-1} \chi_{(-\frac{1}{3}, \frac{1}{3})}$. In particular, we prove that²

$$E_{p,\tau}^{\check{\chi}} = D_{p,\tau}.$$

Moreover, in Proposition 11 we clarify the connection between the Stockwell coefficients and the value of the S-transform with window $\check{\chi}$.

Let φ be an admissible window, we introduce the basis

$$E_{p,\tau}^\varphi, \tag{1.8}$$

such that³

$$(S_\varphi E_{p,\tau}^\varphi)(b, \nu(p)) = D_{p,\tau}(b), \tag{1.9}$$

where $\nu(p)$ is the center of the p -frequency band where the basis $D_{p,\tau}$ in (1.6) is supported. In Section 6, we introduce a fast ($\mathcal{O}(N \log N)$) algorithm to compute the coefficients

² See Remark 3 for the precise statement.

³ Equality (1.9) must be interpreted with care, we refer to Section 6 for the precise statement.

$$(f, E_{p,\tau}^\varphi)_{L^2([0,1])}.$$

Unfortunately, for a general admissible window φ , the basis (1.8) fails to be orthogonal. Nevertheless, under a mild condition on φ , we prove that it forms a frame, which in general is not tight. So, by abstract theory of frames, we obtain that the coefficients in (1.7) are

$$(f, \widetilde{E_{p,\tau}^\varphi})_{L^2([0,1])},$$

where $\widetilde{E_{p,\tau}^\varphi}$ is the dual frame of $E_{p,\tau}^\varphi$.

This paper is organized as follows: in Section 2, we provide a brief survey on the S-transform in the context of time–frequency analysis. In particular, we point out the similarities and the differences between Fourier transform, short-time Fourier transform and wavelet transform. In Section 3, we prove that (1.6) is a basis of $L^2([0, 1])$ and we highlight its time–frequency local properties. In Section 4, we decompose the Stockwell transform with a general window using (1.6). Moreover, we determine the explicit expression of $(S_\varphi D_{p,\tau})$. In Section 5, we provide a discretization of the S-transform. In Section 6, we determine the basis (1.8) adapted to a general admissible window φ . We propose an algorithm which evaluates the coefficients related to the basis (1.8) of computational complexity $\mathcal{O}(N \log N)$. This algorithm extends the one proposed by Y. Wang and J. Orchard in [33].

2. A brief survey on the S-transform

In many practical applications it is important to analyze signals, *i.e.* extracting the time–frequency content of a signal. Given a signal f in $L^2(\mathbb{R})$, we can precisely extract its frequency content using the Fourier transform F

$$\widehat{f}(\xi) = (F f)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t\xi} f(t) dt, \quad \xi \in \mathbb{R}.$$

Unfortunately, due to uncertainty principle, it is impossible to retain at the same time precise time–frequency information. In the past years, many techniques arose trying to deal with the uncertainty principle in order to obtain a *sufficiently good* time–frequency representation of a signal. The short-time Fourier transform

$$(\text{STFT}_\varphi f)(b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t\xi} f(t) \overline{\varphi(t-b)} dt, \quad b, \xi \in \mathbb{R}$$

is one of the standard tools. Loosely speaking, taking the short-time Fourier transform of a signal f at a certain time b is like taking the Fourier transform of the signal f cut by a window function φ centered in b , see for example [13,17]. It is possible to invert the short-time Fourier transform using the following theorem.

Theorem 1. *Let f be a signal in $L^2(\mathbb{R})$ and φ a window in $L^2(\mathbb{R})$. Then*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} (\text{STFT}_\varphi f)(b, \xi) db, \quad \xi \in \mathbb{R}.$$

Notice that the width of the analyzing window remains fixed. Due to the Nyquist sampling theorem, it would be natural to consider a window whose width depends on the analyzed frequency. To accomplish this task, in [29], the S-transform S_g was introduced as

$$(S_g f)(b, \xi) = (2\pi)^{-1/2} |\xi| \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) e^{-(t-b)^2 \xi^2 / 2} dt, \quad b, \xi \in \mathbb{R}. \tag{2.1}$$

Notice that the width of the Gaussian window $e^{-(t-b)^2 \xi^2 / 2}$ shrinks as the analyzed frequency increases, providing a better time-localization for high frequencies. It is possible to rewrite the Stockwell transform with respect to the Fourier transform of the signal f as

$$(S_g f)(b, \xi) = \int_{\mathbb{R}} e^{2\pi i \zeta b} \widehat{f}(\zeta + \xi) e^{-\frac{2\pi^2 \zeta^2}{\xi^2}} d\zeta, \quad b, \xi \in \mathbb{R}, \xi \neq 0. \tag{2.2}$$

In [29] it has been stated an inversion formula similar to Theorem 1.

Theorem 2. *Let f be a signal in $L^2(\mathbb{R})$. Then*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} (S_g f)(b, \xi) db, \quad \xi \in \mathbb{R}.$$

Many extensions of this transform have been suggested in the last years. See for example [10,18,19,34,35]. We here recall the one introduced in [34].

Definition 1. Let f be a signal in $L^2(\mathbb{R})$ and let φ be a window function in $L^2(\mathbb{R})$. Then, we call

$$(S_\varphi f)(b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) |\xi| \overline{\varphi(\xi(t-b))} dt, \quad b, \xi \in \mathbb{R} \tag{2.3}$$

the S_φ -transform of the signal f with respect to the window φ .

It is possible to recover the original definition (2.1) taking φ to be the Gaussian window $\varphi(t) = e^{-t^2/2}$. The S-transform has been recently extended to the multi-dimensional case by the second author [26]. Theorem 2 still holds for the S-transform (2.3).

See [4,7,12,16,20,22,24,36] for some applications of the S-transform to signal processing.

Heuristically, we can think at the S-transform as a short-time Fourier transform in which the width of the analyzing window varies with respect to the analyzed frequency. Therefore, the S-transform can also be interpreted as a particular non-stationary Gabor transform, see [1].

We can give an equivalent definition of the S-transform using the following proposition.

Proposition 3. *Let f be a signal in $L^2(\mathbb{R})$ and let φ be a window in $L^2(\mathbb{R})$. Then*

$$(S_\varphi f)(b, \xi) = e^{-2\pi i b \xi} \left(F_{\zeta \rightarrow b}^{-1} f_\xi \right)(b), \quad b, \xi \in \mathbb{R}, \xi \neq 0,$$

where

$$f_\xi(\zeta) = \widehat{f}(\zeta) \overline{\widehat{\varphi}\left(\frac{\zeta - \xi}{\xi}\right)}, \quad \zeta \in \mathbb{R}, \xi \neq 0.$$

The following inversion formula has been proven in [34].

Theorem 4. *Let φ be a function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that*

$$c_\varphi = \int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi + 1|} < \infty.$$

We say that φ is an admissible window for the S-transform and we call c_φ the admissibility constant. Then

$$c_\varphi (f, f')_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} (S_\varphi f)(b, \xi) \overline{(S_\varphi f')(b, \xi)} db \frac{d\xi}{|\xi|},$$

for all f and f' in $L^2(\mathbb{R})$.

Notice that [Theorem 2](#) and [Theorem 4](#) still hold in the multi-dimensional case (see [\[25,26\]](#)).

At this point, it is useful to recall the wavelet transform $W_\varphi f$ of a signal f in $L^2(\mathbb{R})$ with respect to the window φ

$$(W_\varphi f)(b, a) = \int_{\mathbb{R}} f(t) |a|^{-1/2} \overline{\varphi(a^{-1}(t-b))} dt, \quad \forall b, a \in \mathbb{R}.$$

See for example [\[3,9,23\]](#) for details on wavelet analysis and filter banks.

Theorem 5. Let φ be a window in $L^2(\mathbb{R})$ such that

$$c_\varphi = \int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty.$$

We say that φ is an admissible wavelet and we call c_φ the admissibility constant. Then

$$c_\varphi (f, f')_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(b, a) \overline{(W_\varphi f')(b, a)} db \frac{da}{a^2},$$

for all f and f' in $L^2(\mathbb{R})$.

Notice the similarities between [Theorem 4](#) and [Theorem 5](#). This follows from a deep connection among Stockwell transform, short-time Fourier transform and wavelet transform. In fact, these transforms are related to the affine Weyl–Heisenberg group studied in [\[21\]](#). This connection has been highlighted in the multi-dimensional case by the second author in [\[25\]](#). In [\[15,30\]](#), the connections between Stockwell transform and wavelet transform are pointed out. The affine Weyl–Heisenberg group is also connected to the definition of α -modulation spaces, see [\[2,8,14\]](#), which represents, at the level of coorbit theory, a sort of interpolation between Modulation spaces and Besov spaces. A different group approach to the Stockwell transform has been studied in [\[6\]](#).

3. A time–frequency localized basis

In this section, we prove that the system of functions [\(1.6\)](#), proposed by R.G. Stockwell in [\[27\]](#), is indeed an orthonormal basis of $L^2([0, 1])$.

For $p = 0$, we define

$$\nu(0) = 0, \quad \beta(0) = 1, \quad \tau(0) = 0,$$

for $p = 1$

$$\nu(1) = 1, \quad \beta(1) = 1, \quad \tau(1) = 0,$$

for all $p \geq 2$

$$\nu(p) = 2^{p-1} + 2^{p-2}, \quad \beta(p) = 2^{p-1}, \quad \tau(p) = 0, \dots, \beta(p) - 1.$$

Setting, for each p , the p -frequency band

$$[\beta(p), 2\beta(p) - 1] = \left[\nu(p) - \frac{\beta(p)}{2}, \nu(p) + \frac{\beta(p)}{2} - 1 \right],$$

we obtain a partition of \mathbb{N} ; notice that $\nu(p)$ is the center of each p -frequency band. We recall here the definition of the so-called DOST functions, introduced in [27]:

$$\begin{aligned} D_0(t) &= 1, & t \in \mathbb{R}, \\ D_1(t) &= e^{2\pi i t}, & t \in \mathbb{R}, \end{aligned}$$

and

$$D_p = \{D_{p,\tau}(t)\}_{\tau=0,\dots,\beta(p)-1}, \quad t \in \mathbb{R},$$

where

$$D_{p,\tau}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{f=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} e^{2\pi i f t} e^{-2\pi i f \tau / \beta(p)}, \quad t \in \mathbb{R}.$$

For all negative integers p , we set

$$D_{p,\tau}(t) = \overline{D_{-p,\tau}(t)}, \quad \tau = 0, \dots, \beta(|p|) - 1.$$

For each $p \in \mathbb{N}$, $\nu(-p) = -\nu(p)$ and $\beta(-p) = \beta(p)$. In the sequel we call

$$\bigcup_{p \in \mathbb{Z}} D_p \tag{3.1}$$

Stockwell basis.

Notice that, in the original paper [27], each $D_{p,\tau}$ had a multiplicative factor $e^{\tau\pi i}$. Since this factor is not crucial in proving that (3.1) is a basis of $L^2([0, 1])$, we have decided to drop it. In (5.17), we clarify the role of this multiplicative factor. In Fig. 1 and Fig. 2 we draw the DOST basis functions without this multiplicative factor. In Fig. 2 notice that, with our choice, these functions are self-similar in each p -band, in contrast to the ones defined in [27]. Moreover, we have slightly changed the notation in the frequency domain. The k th element of the Fourier basis is $e^{2\pi i k t}$, while, in the original paper, the k th element is $e^{-2\pi i k t}$. The convention we adopt seems closer to the standard Fourier analysis.

Theorem 6. $\bigcup_{p \in \mathbb{Z}} D_p$ is an orthonormal basis of $L^2([0, 1])$.

Proof. In the sequel, we consider positive p . For negative p , all results hold true using the adjoint property. We recall that $\{e^{2\pi i k t}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of $L^2([0, 1])$ and we notice that $D_{p,\tau}(t)$ is a finite linear combination of $e^{2\pi i k t}$ with k in the p -frequency band

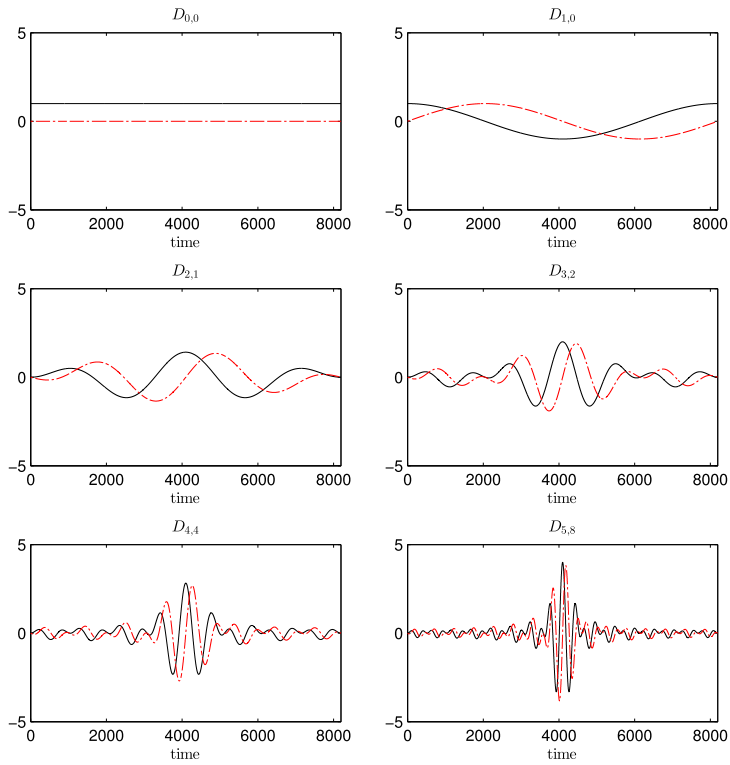


Fig. 1. DOST basis functions in increasing frequency p -bands. Black line = real, red line = imaginary. See Fig. 2 in [27] for a comparison.

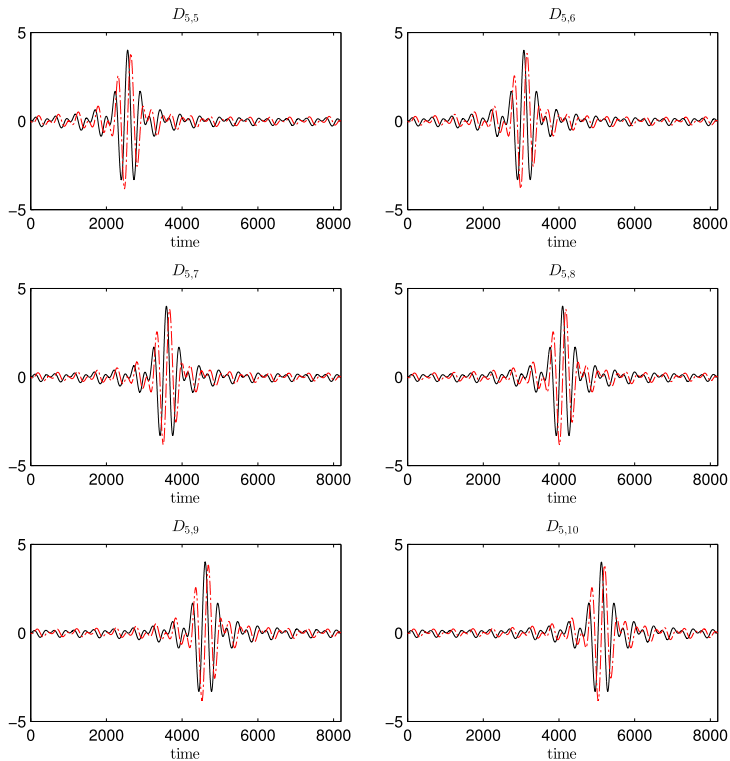


Fig. 2. DOST basis functions in the same p -band ($p = 5$). Black line = real, red line = imaginary. See Fig. 1 in [27] for a comparison.

$$I_p = \left[\nu(p) - \frac{\beta(p)}{2}, \nu(p) + \frac{\beta(p)}{2} - 1 \right].$$

Hence, we can conclude that

$$(D_{p,\tau}, D_{p',\tau'})_{L^2([0,1])} = 0, \quad \text{if } p \neq p', \quad \forall \tau, \tau',$$

since the p -band and the p' -band are disjoint. So, we can focus on the case $p = p'$. The proof is divided into three steps.

Step I - $\|D_{p,\tau}\|_{L^2([0,1])} = 1.$

Consider the inner product

$$\begin{aligned} \|D_{p,\tau}\|_{L^2([0,1])}^2 &= (D_{p,\tau}, D_{p,\tau})_{L^2([0,1])} \\ &= \frac{1}{\beta(p)} \int_0^1 \left(\sum_{f=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} e^{2\pi i f t} e^{-2\pi i f \tau / \beta(p)} \right) \left(\sum_{f'=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} e^{-2\pi i f' t} e^{2\pi i f' \tau / \beta(p)} \right) dt. \end{aligned}$$

Since $\{e^{2\pi i k t}\}_{k \in \mathbb{Z}}$ is an orthonormal basis,

$$\|D_{p,\tau}\|_{L^2([0,1])}^2 = \frac{1}{\beta(p)} \sum_{f=\nu(p)-\beta(p)/2}^{f=\nu(p)+\beta(p)/2-1} \int_0^1 1 dt = 1.$$

Step II - $\bigcup_{p \in \mathbb{Z}} D_p$ is an orthonormal set.

If $p \neq p'$ the L^2 -scalar product vanishes, so we can suppose $p = p'$. It is convenient to consider $j = f - \beta(p)$.

$$\begin{aligned} D_{p,\tau}(t) &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)t} e^{-2\pi i (\beta(p)+j)\tau / \beta(p)} \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)t} e^{-2\pi i \tau j / \beta(p)}. \end{aligned} \tag{3.2}$$

The orthonormality of the Fourier basis implies

$$(D_{p,\tau}, D_{p,\tau'})_{L^2([0,1])} = \frac{1}{\beta(p)} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\tau' - \tau)j / \beta(p)}. \tag{3.3}$$

Now, we need the following lemma.

Lemma 1. *Let $k \in \mathbb{N} \setminus \{0\}$. Then*

$$\sum_{j=0}^{2^k-1} e^{2\pi i j m / 2^k} = 0, \quad m = \pm 1, \dots, \pm(2^k - 1). \tag{3.4}$$

Proof. Notice that (3.4) is a truncated geometric series with ratio $e^{2\pi i m / 2^k}$. Therefore, the well known formula for geometric progression implies that

$$\sum_{j=0}^{2^k-1} e^{2\pi i j m / 2^k} = \frac{1 - e^{2\pi i m 2^k / 2^k}}{1 - e^{2\pi i m / 2^k}} = 0.$$

Since $m = \pm 1, \dots, \pm(2^k - 1)$, the denominator in the above equation never vanishes. \square

Let $(\tau' - \tau) = m$ in (3.4), then Lemma 1 implies that

$$(D_{p,\tau}, D_{p',\tau'})_{L^2([0,1])} = \delta_0(p - p')\delta_0(\tau - \tau'),$$

i.e. $\bigcup_{p \in \mathbb{Z}} D_p$ is an orthonormal set.

Step III - $\bigcup_{p \in \mathbb{Z}} D_p$ is a basis of $L^2([0, 1])$.

Notice that

$$D_p \subseteq \text{span}\{e^{2\pi i kt}\}_{k \in [\beta(p), 2\beta(p)-1]}.$$

Hence, to prove the assertion it is sufficient to show that the elements of the set $\{D_{p,\tau}\}_{\tau=0, \dots, \beta(p)-1}$ are a basis of $\text{span}\{e^{2\pi i kt}\}_{k \in [\beta(p), 2\beta(p)-1]}$. Since we deal with finite dimensional vector spaces, we prove that the functions $\{D_{p,\tau}\}_{\tau=0}^{\beta(p)-1}$ are linearly independent; that is

$$\sum_{\tau=0}^{\beta(p)-1} c_\tau D_{p,\tau} = 0 \implies c_\tau = 0, \quad \forall \tau = 0, \dots, \beta(p) - 1. \tag{3.5}$$

Since $\{e^{2\pi i (\beta(p)+j)t}\}_{j=0, \dots, \beta(p)-1}$ is a basis, we can consider the projection of (3.5) on each term $\{e^{2\pi i (\beta(p)+j)t}\}_{j=0, \dots, \beta(p)-1}$ of the Fourier basis. We obtain the system

$$\sum_{\tau=0}^{\beta(p)-1} c_\tau e^{-2\pi i \tau j / \beta(p)} = 0, \quad j = 0, \dots, \beta(p) - 1. \tag{3.6}$$

Notice that (3.6) can be written as the linear system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-2\pi i / \beta(p)} & \dots & e^{-2\pi i (\beta(p)-1) / \beta(p)} \\ 1 & e^{-2\pi i 2 / \beta(p)} & \dots & e^{-2\pi i 2(\beta(p)-1) / \beta(p)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-2\pi i (\beta(p)-1) / \beta(p)} & \dots & e^{-2\pi i (\beta(p)-1)^2 / \beta(p)} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{\beta(p)-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.7}$$

The square matrix in (3.7) is a Vandermonde matrix with entries $\{e^{-2\pi i l / \beta(p)}\}_{l=0}^{\beta(p)-1}$. Since the entries are all distinct the determinant of the Vandermonde matrix is non-zero and the unique solution of the linear system (3.7) is the zero vector. That is the functions $\{D_{p,\tau}\}_{\tau=0}^{\beta(p)-1}$ are linear independent. \square

Lemma 1 implies the following corollary.

Corollary 1. For each $p \in \mathbb{Z}$ and each $\tau, \tau' = 0, \dots, \beta(|p|) - 1$ we have

$$D_{p,\tau} \left(\frac{\tau'}{\beta(p)} \right) = \sqrt{\beta(p)} \delta_0(\tau' - \tau).$$

Proof. Let us suppose p positive. Then

$$\begin{aligned} D_{p,\tau} \left(\frac{\tau'}{\beta(p)} \right) &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j) \left(\frac{\tau'-\tau}{\beta(p)} \right)} \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i j(\tau'-\tau)/\beta(p)}. \end{aligned}$$

Letting $(\tau' - \tau) = m$, we apply Lemma 1 and we obtain the assertion. For negative p we use the adjoint property. \square

The DOST functions are not dilations nor translations of a single function. Nevertheless, for each p ,

$$D_p(t) = \left\{ \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)(t-\tau/\beta(p))} \right\}_{\tau=0,\dots,\beta(p)-1}$$

is formed by translations of $\tau/\beta(p)$ of the same function. Roughly speaking, we can state that the DOST basis is not self similar globally, but it is self similar in each band, see Fig. 2. Hence, the S-transform in this setting appears different from the wavelet transform because the mother wavelet changes as the frequencies increases, in contrast to the usual formulation.

R.G. Stockwell proposed this basis because it is an efficient compromise between frequency localization in low frequencies and time localization for high frequencies. The price to pay is that, on one hand, for high frequencies, we do not have a precise frequency localization, but just a localization in a certain band, which is wider as the frequency increases and, on the other hand, in low frequencies, we lose time localization. In fact, for high frequencies, the basis $D_{p,\tau}$ are, in large sense, local at $t = \tau/\beta(p)$. It is not true that $D_{p,\tau}$ has compact support in time, but the energy is concentrated near the point $t = \tau/\beta(p)$. We prove that the basis functions $D_{p,\tau}$ are 0.85-concentrated in the neighborhood

$$I_{p,\tau} = \left[\frac{\tau}{\beta(p)} - \frac{1}{2\beta(p)}, \frac{\tau}{\beta(p)} + \frac{1}{2\beta(p)} \right],$$

in the sense of the Donoho–Stark Theorem [11,5].

Proposition 7. For each $D_{p,\tau}(t)$ we have

$$\|D_{p,\tau}\|_{L^2(I_{p,\tau})} = \left(\int_{\frac{2\tau-1}{2\beta(p)}}^{\frac{2\tau+1}{2\beta(p)}} |D_{p,\tau}|^2 dt \right)^{1/2} > 0,85,$$

i.e. the L^2 -norm is concentrated in the interval

$$I_{p,\tau} = \left[\frac{\tau}{\beta(p)} - \frac{1}{2\beta(p)}, \frac{\tau}{\beta(p)} + \frac{1}{2\beta(p)} \right].$$

Since $\|D_{p,\tau}\| = 1$, we can also state that the L^2 -norm of $D_{p,\tau}$ is less than 0,15 out of $I_{p,\tau}$. For $\tau = 0$, $I_{p,0}$ must be considered as an interval in circle, that is

$$I_{p,0} = \left[0, \frac{1}{2\beta(p)} \right) \cup \left(1 - \frac{1}{2\beta(p)}, 1 \right].$$

Proof. Since in each p -band the basis functions are a translation of $\tau/\beta(p)$ of the same function, we can prove the property for a fixed τ . For simplicity, we consider $\tau = 0$. In order to take in account just one integral, we extend by periodicity the function for negative t and we evaluate

$$\int_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} |D_{p,\tau}(t)|^2 dt. \tag{3.8}$$

Notice that

$$\begin{aligned} |D_{p,0}|^2 &= D_{p,0}(t) \cdot \overline{D_{p,0}(t)} \\ &= \frac{1}{\beta(p)} \left(\sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)t} \right) \cdot \left(\sum_{k=0}^{\beta(p)-1} e^{-2\pi i (\beta(p)+k)t} \right) \\ &= \frac{1}{\beta(p)} \sum_{m=-\beta(p)+1}^{\beta(p)-1} (\beta(p) - |m|) e^{2\pi i m t}. \end{aligned} \tag{3.9}$$

Equation (3.9) can be proven by induction on the size of the band. Writing (3.9) in terms of cosine and sine we obtain

$$\begin{aligned} |D_{p,0}|^2 &= \frac{1}{\beta(p)} \sum_{m=-\beta(p)+1}^{\beta(p)-1} (\beta(p) - |m|) (\cos(2\pi m t) + i \sin(2\pi m t)) \\ &= 1 + \frac{1}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) ((\cos(2\pi m t) + \cos(-2\pi m t)) + i (\sin(2\pi m t) + \sin(-2\pi m t))) \\ &= 1 + \frac{2}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \cos(2\pi m t). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} |D_{p,0}(t)|^2 dt &= \int_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} dt + \frac{2}{\beta(p)} \int_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \cos(2\pi m t) dt \\ &= \frac{1}{\beta(p)} + \frac{2}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \frac{\sin(2\pi m t)}{2\pi m} \Big|_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} \\ &= \frac{1}{\beta(p)} + \frac{4}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \frac{\sin\left(\frac{2\pi m}{2\beta(p)}\right)}{2\pi m}. \end{aligned}$$

By the Maclaurin expansion of $\sin(x)$,

$$\int_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} |D_{p,0}|^2 dt = \frac{1}{\beta(p)} + \frac{4}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \left(\frac{2\pi m \frac{1}{2\beta(p)} + R_m(\eta)}{2\pi m} \right),$$

where $R_m(\eta)$ is the Lagrange rest. Using Gauss summation formula we obtain

$$\begin{aligned} \int_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} |D_{p,0}(t)|^2 dt &\cong \frac{1}{\beta(p)} + \frac{2}{\beta^2(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \\ &= \frac{1}{\beta(p)} + \frac{2}{\beta^2(p)} \left(\beta(p)(\beta(p) - 1) - \frac{1}{2}\beta(p)(\beta(p) - 1) \right) \\ &= \frac{1}{\beta(p)} + \frac{1}{\beta(p)}(\beta(p) - 1) = 1. \end{aligned}$$

We have to take in account the rests $R_m(\eta)$. Since

$$\sup \left| \frac{d^3}{dt^3} [\sin(2\pi mt)] \right| = (2\pi m)^3,$$

we can conclude that

$$\begin{aligned} &\left| \frac{4}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \frac{R_m(\eta)}{2\pi m} \right| \\ &\leq \frac{4}{\beta(p)} \sum_{m=1}^{\beta(p)-1} \frac{(\beta(p) - m)}{2\pi m} \frac{(2\pi m)^3}{6(2\beta(p))^3} \\ &\leq \frac{\pi^2}{3\beta^4(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m)m^2 \\ &\leq \frac{\pi^2}{3\beta^4(p)} \left(\frac{\beta(p)^2}{6}(\beta(p) - 1)(2\beta(p) - 1) - \frac{\beta(p)^2}{4}(\beta(p) - 1)^2 \right) \\ &\leq \frac{\pi^2(\beta(p) - 1)}{3\beta^2(p)} \left(\frac{\beta(p)}{3} - \frac{1}{6} - \frac{\beta(p)}{4} + \frac{1}{4} \right) \\ &\leq \frac{\pi^2}{36} \frac{\beta(p)^2 - 1}{\beta(p)^2} \\ &\leq \frac{\pi^2}{36} < 0,275. \end{aligned}$$

Hence, finally

$$\left(\int_{-\frac{1}{2\beta(p)}}^{\frac{1}{2\beta(p)}} |D_{p,0}(t)|^2 dt \right)^{1/2} \geq (1 - 0,275)^{1/2} \geq \sqrt{0,725} > 0,85. \quad \square$$

4. Diagonalization of the S-transform

In this section, for the sake of clarity, we write S_φ -transform instead of S-transform to emphasize the window dependence. We focus our attention to $L^2([0,1])$. Using Fourier series, it is well known that if $f \in L^2([0,1])$, then

$$f(t) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k t}, \quad \text{a.e.,}$$

and

$$\|f\|_{L^2([0,1])} = \left(\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \right)^{\frac{1}{2}}.$$

We define the Hilbert space $(Y, (\cdot, \cdot)_Y, \| \cdot \|_Y)$:

$$Y = \left\{ \sum_{k \in \mathbb{Z}} c_k(\xi) e^{2\pi i (k-\xi)b} \mid c_k(\xi) \in L^2 \left(\mathbb{R}, \frac{1}{|\xi|} \right), \text{ and } \sum_{k \in \mathbb{Z}} \|c_k\|_{L^2 \left(\mathbb{R}, \frac{1}{|\xi|} \right)}^2 < \infty \right\},$$

$$(g, g')_Y = \int_0^1 \int_{\mathbb{R}} g(b, \xi) \overline{g'(b, \xi)} \frac{d\xi}{|\xi|} db, \quad g, g' \in Y,$$

$$\|g\|_Y = \sqrt{(g, g)_Y} = \left(\sum_{k \in \mathbb{Z}} \|g_k\|_{L^2 \left(\mathbb{R}, \frac{1}{|\xi|} \right)}^2 \right)^{\frac{1}{2}}, \quad g(b, \xi) = \sum_{k \in \mathbb{Z}} g_k(\xi) e^{2\pi i (k-\xi)b}, \quad \text{a.e..}$$

In view of [Theorem 4](#), we introduce $(Z, (\cdot, \cdot)_Z, \| \cdot \|_Z)$ the Hilbert space of admissible windows:

$$Z = \left\{ \varphi \in \mathcal{S}'(\mathbb{R}) \mid \int |\widehat{\varphi}(\xi)|^2 \frac{d\xi}{|1+\xi|} < \infty \right\}, \tag{4.1}$$

$$(\varphi, \varphi')_Z = \int \widehat{\varphi}(\xi) \overline{\widehat{\varphi}'(\xi)} \frac{d\xi}{|1+\xi|}, \quad \varphi, \varphi' \in Z,$$

$$\|\varphi\|_Z = \sqrt{(\varphi, \varphi)_Z} = \left(\int |\widehat{\varphi}(\xi)|^2 \frac{d\xi}{|1+\xi|} \right)^{1/2}.$$

Theorem 8. *We define*

$$S : L^2([0, 1]) \times (Z \cap \mathcal{S}'(\mathbb{R})) \longrightarrow Y$$

$$(f, \varphi) = \left(\sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k t}, \varphi \right) \longmapsto \sum_{k \in \mathbb{Z}} \hat{f}(k) (S_\varphi(e^{2\pi i k \cdot}))(b, \xi)$$

where, in view of [Proposition 3](#), we set

$$(S_\varphi e^{2\pi i k \cdot})(b, \xi) = e^{-2\pi i b \xi} F_{\zeta \mapsto b}^{-1} \left(\overline{\widehat{\varphi} \left(\frac{\zeta - \xi}{\xi} \right)} \delta_k(\zeta) \right) (b).$$

Then $S : L^2([0, 1]) \times (Z \cap \mathcal{S}'(\mathbb{R})) \longrightarrow Y$ is continuous.

Proof. We start considering $S(e^{2\pi i k \cdot}, \varphi) = (S_\varphi e^{2\pi i k \cdot})$. By definition,

$$(S_\varphi e^{2\pi i k \cdot})(b, \xi) = e^{-2\pi i b \xi} F_{\zeta \mapsto b}^{-1} \left(\overline{\widehat{\varphi} \left(\frac{\zeta - \xi}{\xi} \right)} \delta_k(\zeta) \right) (b)$$

$$= e^{-2\pi i b \xi} F_{\zeta \mapsto b}^{-1} \left(\overline{\widehat{\varphi} \left(\frac{k - \xi}{\xi} \right)} \delta_k(\zeta) \right) (b)$$

$$\begin{aligned}
 &= e^{-2\pi i b\xi} \widehat{\varphi} \left(\frac{k-\xi}{\xi} \right) e^{2\pi i kb} \\
 &= e^{2\pi i b(k-\xi)} \widehat{\varphi} \left(\frac{k-\xi}{\xi} \right).
 \end{aligned}$$

We observe that

$$\begin{aligned}
 \left\| \widehat{\varphi} \left(\frac{k-\cdot}{\cdot} \right) \right\|_{L^2([0,1], \frac{1}{|\xi|})}^2 &= \int_{\mathbb{R}} \left| \widehat{\varphi} \left(\frac{k-\xi}{\xi} \right) \right|^2 \frac{1}{|\xi|} d\xi \\
 &= \int_{\mathbb{R}} \left| \widehat{\varphi}(\omega-1) \right|^2 \frac{|\omega|}{|k|} \frac{|k|}{|\omega|^2} d\omega \\
 &= \int_{\mathbb{R}} \left| \widehat{\varphi}(w) \right|^2 \frac{dw}{|w+1|} = \|\varphi\|_Z^2.
 \end{aligned} \tag{4.2}$$

Therefore,

$$\|\mathbb{S}(e^{2\pi i k \cdot}, \varphi)\|_Y^2 = \|\mathbb{S}_\varphi e^{2\pi i k \cdot}\|_Y^2 = \|\varphi\|_Z^2.$$

The functions $\{e^{2\pi i kt}\}_{k \in \mathbb{Z}}$ are orthonormal in $L^2([0,1])$. Notice that

$$\begin{aligned}
 (\mathbb{S}_\varphi e^{2\pi i k \cdot}, \mathbb{S}_\varphi e^{2\pi i k' \cdot})_Y &= \int_0^1 e^{2\pi i (k-k')b} db \int_{\mathbb{R}} \widehat{\varphi} \left(\frac{k-\xi}{\xi} \right) \widehat{\varphi} \left(\frac{k'-\xi}{\xi} \right) \frac{d\xi}{|\xi|} \\
 &= \|\varphi\|_Z^2 \delta_0(k-k').
 \end{aligned} \tag{4.3}$$

Using the definition of \mathbb{S} and Eq. (4.3), we conclude that if

$$f(t) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{2\pi i kt}, \quad \text{a.e.,}$$

then

$$\begin{aligned}
 \|(\mathbb{S}_\varphi f)(\cdot, \cdot)\|_Y^2 &= (\mathbb{S}_\varphi f, \mathbb{S}_\varphi f)_Y \\
 &= \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \left(\widehat{f}(k) \mathbb{S}_\varphi e^{2\pi i k \cdot}, \widehat{f}(k') \mathbb{S}_\varphi e^{2\pi i k' \cdot} \right)_Y \\
 &= \sum_{k \in \mathbb{Z}} \left(\widehat{f}(k) \mathbb{S}_\varphi e^{2\pi i k \cdot}, \widehat{f}(k) \mathbb{S}_\varphi e^{2\pi i k \cdot} \right)_Y \\
 &= \sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2 \|\mathbb{S}_\varphi e^{2\pi i k \cdot}\|_Y^2 \\
 &= \|\varphi\|_Z^2 \|f\|_{L^2([0,1])}^2.
 \end{aligned}$$

Therefore, $\mathbb{S} : L^2([0,1]) \times (Z \times \mathcal{S}(\mathbb{R})) \rightarrow Y$ is a continuous operator. \square

Lemma 2. Let $\mathbb{S} : L^2([0,1]) \times (Z \cap \mathcal{S}(\mathbb{R})) \rightarrow Y$ defined as in Theorem 8. Then, since $\mathcal{S}(\mathbb{R}) \cap Z$ is dense in Z , we can extend by continuity \mathbb{S} to the whole of Z .

Remark 1. Theorem 8 is the discrete counterpart of Theorem 5 in the case of periodic functions.

In Section 3, we proved that the DOST functions form an orthonormal basis of $L^2([0, 1])$. Let us assume that φ belongs to Z defined in (4.1). Then, by Theorem 8, $S_\varphi : L^2([0, 1]) \rightarrow Y$ is continuous. So, we can write

$$\begin{aligned} (S_\varphi f) &= \left(S_\varphi \sum (f, D_{p,\tau})_{L^2([0,1])} D_{p,\tau} \right) \\ &= \sum (f, D_{p,\tau})_{L^2([0,1])} (S_\varphi D_{p,\tau}) \\ &= \sum f_{p,\tau} (S_\varphi D_{p,\tau}), \end{aligned} \tag{4.4}$$

where

$$f_{p,\tau} = (f, D_{p,\tau})_{L^2([0,1])}$$

and the sum in (4.4) is over all $D_{p,\tau}$ functions. Hence, in order to understand the S_φ -transform of a general function $f \in L^2([0, 1])$, it is sufficient to evaluate the coefficients $f_{p,\tau}$ and determine once for all the S_φ -transform of $D_{p,\tau}$.

Notice that, for $p > 0$,

$$\begin{aligned} D_{p,\tau}(t) &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)(t-\tau/\beta(p))} \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} T_{-\tau/\beta(p)} M_{\beta(p)+j} \mathbb{1}(t). \end{aligned}$$

Hence, we can write

$$\begin{aligned} (F D_{p,\tau})(\xi) &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} (F T_{-\tau/\beta(p)} M_{(\beta(p)+j)} \mathbb{1})(\xi) \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} (M_{-\tau/\beta(p)} T_{-\beta(p)-j} F \mathbb{1})(\xi) \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} (M_{-\tau/\beta(p)} T_{-\beta(p)-j} \delta_0)(\xi) \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{-2\pi i \frac{\tau}{\beta(p)} \xi} \delta_0(\xi - \beta(p) - j) \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{-2\pi i (\beta(p)+j) \frac{\tau}{\beta(p)}} \delta_{\beta(p)+j}(\xi). \end{aligned} \tag{4.5}$$

Let us compute the S_φ -transform of a basis function $D_{p,\tau}$ with a general window φ belonging to Z . We assume $\hat{\varphi}$ continuous. By Theorem 8 and Eq. (4.5), we obtain

$$\begin{aligned}
 & e^{2\pi i b \xi} (S_\varphi D_{p,\tau})(b, \xi) \\
 &= F_{\zeta \rightarrow b}^{-1} \left(\overline{\widehat{\varphi} \left(\frac{\zeta - \xi}{\xi} \right)} (F D_{p,\tau})(\zeta) \right) (b) \\
 &= F_{\zeta \rightarrow b}^{-1} \left(\sum_{j=0}^{\beta(p)-1} \frac{e^{-2\pi i (\beta(p)+j)\tau/\beta(p)}}{\sqrt{\beta(p)}} \overline{\widehat{\varphi} \left(\frac{\zeta - \xi}{\xi} \right)} \delta_{(\beta(p)+j)}(\zeta) \right) (b) \\
 &= F_{\zeta \rightarrow b}^{-1} \left(\sum_{j=0}^{\beta(p)-1} \frac{e^{-2\pi i (\beta(p)+j)\tau/\beta(p)}}{\sqrt{\beta(p)}} \overline{\widehat{\varphi} \left(\frac{\beta(p) + j - \xi}{\xi} \right)} \delta_{(\beta(p)+j)}(\zeta) \right) (b) \\
 &= \left(\sum_{j=0}^{\beta(p)-1} \frac{e^{-2\pi i (\beta(p)+j)\tau/\beta(p)}}{\sqrt{\beta(p)}} \overline{\widehat{\varphi} \left(\frac{\beta(p) + j - \xi}{\xi} \right)} e^{2\pi i b(\beta(p)+j)} \right) (b). \tag{4.6}
 \end{aligned}$$

We set, for each fixed window φ , $c_{p,j}^\varphi : \mathbb{R} \rightarrow \mathbb{R}$ as

$$c_{p,j}^\varphi(\xi) = \overline{\widehat{\varphi} \left(\frac{\beta(p) + j - \xi}{\xi} \right)}, \quad \xi \neq 0. \tag{4.7}$$

Hence, (4.6) simplifies into

$$(S_\varphi D_{p,\tau})(b, \xi) = e^{-2\pi i b \xi} \left(\sum_{j=0}^{\beta(p)-1} \frac{e^{2\pi i (\beta(p)+j)(b-\tau/\beta(p))}}{\sqrt{\beta(p)}} c_{p,j}^\varphi(\xi) \right), \quad \xi \neq 0. \tag{4.8}$$

Eqs. (4.8) and (4.4) provide an explicit expression of the S_φ -transform of a periodic signal f in terms of its Stockwell coefficients $f_{p,\tau}$. Notice that if $\widehat{\varphi}$ is not a continuous function then Eqs. (4.6), (4.7) and Eq. (4.8) must be understood as a.e. equivalences.

5. Discretization of the S_φ -transform

Let us consider an admissible window and a dyadic decomposition of the frequency domain (see Section 3). We study the S_φ -transform of the periodic signal f at $\xi = \nu(p)$. Some conditions on the window φ are necessary in order to evaluate S_φ -transform punctually.

Assumption 1. Let $\widehat{\varphi}$ be a function in $L^\infty(\mathbb{R})$ such that $\widehat{\varphi}|_{(-\frac{1}{3}, \frac{1}{3})}$ is continuous, and such that

$$\begin{aligned}
 & \widehat{\varphi}(\xi) \neq 0, \quad |\xi| < \frac{1}{3}, \\
 & \widehat{\varphi}(\xi) = 0, \quad |\xi| > \frac{1}{3}, \quad \text{a.e.} \\
 & \lim_{\xi \rightarrow -\frac{1}{3}^+} \widehat{\varphi}(\xi) = c < \infty.
 \end{aligned}$$

Notice that φ belongs to the set of admissible windows Z .

In the sequel we want to evaluate $\widehat{\varphi}$ punctually. So, we need to perform a regularizing procedure.

Lemma 3. *Let φ be an admissible function satisfying Assumption 1. Then it is possible to construct a sequence of continuous functions $\{\varphi_R\}_{R=1}^\infty$ such that⁴*

$$\widehat{\varphi}_R \text{ is continuous,}$$

$$\widehat{\varphi}_R(\xi) \rightarrow \widehat{\varphi}(\xi), \quad \text{punctually,} \tag{5.1}$$

$$\varphi_R(\xi) \rightarrow \varphi(\xi), \quad \text{in the set of admissible windows } Z. \tag{5.2}$$

Moreover, we can suppose that

$$\overline{\widehat{\varphi}_R(\xi)} = 0, \quad \xi \in \mathbb{R} \setminus \left(-\frac{1}{3} - \frac{2}{3\beta(R)}, \frac{1}{3} \right), \tag{5.3}$$

$$\overline{\widehat{\varphi}_R(\xi)} = \widehat{\varphi}(\xi), \quad \xi \in \left(-\frac{1}{3}, \frac{1}{3} - \frac{2}{3\beta(R)} \right). \tag{5.4}$$

Proof. We can consider the smooth function

$$\omega_R(\xi) = \begin{cases} 0, & \xi \in \mathbb{R} \setminus \left(-\frac{1}{3} - \frac{2}{3\beta(R)}, \frac{1}{3} \right), \\ 1, & \xi \in \left(-\frac{1}{3}, \frac{1}{3} - \frac{2}{3\beta(R)} \right). \end{cases}$$

Since φ satisfies Assumption 1, we can define

$$\widetilde{\varphi}(\xi) = \begin{cases} \lim_{\xi \rightarrow -\frac{1}{3}^+} \widehat{\varphi}(\xi) & \xi \leq -\frac{1}{3}, \\ \widehat{\varphi}(\xi), & \xi > -\frac{1}{3}. \end{cases}$$

Then $\varphi_R(t) = F_{\xi \rightarrow t}^{-1}(\omega_R(\xi) \widetilde{\varphi}(\xi))$ has the desired properties. \square

Let φ be an admissible window satisfying Assumption 1 and $\{\varphi_R\}_{R=1}^\infty$ a sequence as in Lemma 3. Then, by Eq. (4.8), we can write

$$(S_{\varphi_R} D_{p',\tau})(b, \nu(p)) = e^{-2\pi i b \nu(p)} \left(\sum_{j=0}^{\beta(p')-1} \frac{e^{2\pi i(\beta(p')+j)(t-\tau/\beta(p'))}}{\sqrt{\beta(p')}} c_{p',j}^{\varphi_R}(\nu(p)) \right). \tag{5.5}$$

Clearly, it is crucial to understand the values $c_{p',j}^{\varphi_R}(\nu(p))$, which depend on the window φ only if $|p| \leq R$ and $|p'| \leq R$.

Proposition 9. *Let φ be an admissible window satisfying Assumption 1 and $\{\varphi_R\}_{R=1}^\infty$ be the associated sequence defined in Lemma 3. Then*

$$c_{p',j}^{\varphi_R}(\nu(p)) = 0, \quad \forall j = 0, \dots, \beta(p') - 1 \quad \text{if } p' \neq p, |p| \leq R, |p'| \leq R. \tag{5.6}$$

Proof. We restrict ourselves to positive p' . For $p' < 0$, it suffices to consider the adjoint.

Let $|p| < R$, as in (5.6). In view of the properties of φ_R , in particular (5.3), the condition

⁴ Notice that, if φ satisfies (5.3) and (5.4), then (5.1) implies (5.2) by means of Lebesgue's Convergence Theorem.

$$\left(\frac{\beta(p') + j}{\nu(p)}\right) - 1 \notin \left(-\frac{1}{3} - \frac{2}{3\beta(R)}, \frac{1}{3}\right), \quad p \neq p', j = 0, \dots, \beta(p') - 1 \tag{5.7}$$

implies relation (5.6). If p is negative, then $\nu(p) < 0$ and

$$\left(\frac{\beta(p') + j}{\nu(p)}\right) - 1 < -1 \leq -\frac{1}{3} - \frac{2}{3\beta(R)},$$

hence (5.7) is fulfilled for all $j = 0, \dots, \beta(p') - 1$.

If p positive, recalling the definition of $\beta(p')$ and $\nu(p')$, condition (5.7) turn into

$$\frac{2}{3} \left(\frac{\beta(p')}{\beta(p)} + \frac{j}{\beta(p)}\right) - 1 \notin \left(-\frac{1}{3} - \frac{2}{3\beta(R)}, \frac{1}{3}\right), \quad j = 0, \dots, \beta(p') - 1, p \neq p'. \tag{5.8}$$

If $p \neq p'$, then we have to consider two cases.

Case I – $p' < p$.

The definition of $\beta(p')$ implies that $\beta(p') \leq \beta(p)/2$. Therefore,

$$\begin{aligned} \frac{2}{3} \left(\frac{\beta(p')}{\beta(p)} + \frac{j}{\beta(p)}\right) - 1 &\leq \frac{2}{3} \left(\frac{1}{2} + \frac{j}{\beta(p)}\right) - 1 \\ &\leq -\frac{2}{3} + \frac{2}{3} \frac{j}{\beta(p)} \leq -\frac{2}{3} + \frac{2}{3} \frac{\beta(p') - 1}{\beta(p)} \leq -\frac{2}{3} + \frac{1}{3} - \frac{2}{3\beta(p)} \leq -\frac{1}{3} - \frac{2}{3\beta(R)}. \end{aligned}$$

Case II – $p' > p$.

We have $\beta(p) \leq \beta(p')/2$, so we can write

$$\frac{2}{3} \left(\frac{\beta(p')}{\beta(p)} + \frac{j}{\beta(p)}\right) - 1 \geq \frac{2}{3} \left(2 + \frac{j}{\beta(p)}\right) - 1 \geq \frac{1}{3} + \frac{2}{3} \frac{j}{\beta(p)} \geq \frac{1}{3}.$$

Thus, (5.8) is fulfilled in both cases. □

Let φ be an admissible window satisfying Assumption 1 and $\{\varphi_R\}$ be as in Lemma 3. Then, by Proposition 9, the expression (5.5) assumes a simplified form since it vanishes for all $p' \neq p$, provided $|p'| \leq R$ and $|p| \leq R$. When $p = p'$ we have

$$(S_{\varphi_R} D_{p,\tau})(b, \nu(p)) = e^{-2\pi i b \nu(p)} \left(\sum_{j=0}^{\beta(p)-1} \frac{e^{2\pi i (\beta(p)+j)(b-\tau/\beta(p))}}{\sqrt{\beta(p)}} c_{p,j}^{\varphi_R}(\nu(p)) \right). \tag{5.9}$$

Assume that $c_{p,j}^{\varphi_R}(\nu(p)) = 1$ for all $j = 0, \dots, \beta(p) - 1$, then, via (5.9)

$$(S_{\varphi_R} D_{p,\tau})(b, \nu(p)) = e^{-2\pi i b \nu(p)} D_{p,\tau}(b). \tag{5.10}$$

In order to extend (5.10) to all $D_{p,\tau}$, we introduce the following proposition.

Proposition 10. Set $\tilde{\chi} = F^{-1} \chi$ be such that

$$\overline{(F \tilde{\chi})}(\xi) = \tilde{\chi}(\xi) = \begin{cases} 0 & \xi \in (-\infty, -\frac{1}{3}] \cup [\frac{1}{3}, +\infty) \\ 1 & \xi \in (-\frac{1}{3}, \frac{1}{3}) \end{cases}. \tag{5.11}$$

Then $\check{\chi}$ satisfies Assumption 1 and

$$(S_{\check{\chi}_R} D_{p,\tau}) (b, \nu(p')) = e^{-2\pi i b \nu(p)} D_{p,\tau}(b) \delta_0(p - p'), \quad \text{for all } |p| \leq R, |p'| \leq R, \quad (5.12)$$

where $\{\check{\chi}_R\}_{R=1}^\infty$ is a sequence converging to $\check{\chi}$ as in Lemma 3.

Proof. It follows from the definition of $c_{p,j}^\varphi$ and by (5.10). \square

In order to extend the punctual evaluation (5.12) to all periodic signals in $L^2([0, 1])$, we need to introduce another regularizing procedure in the frequency domain.

Definition 2. We define the Fourier multiplier

$$T_R : L^2([0, 1]) \rightarrow L^2([0, 1]) \quad (5.13)$$

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i kt} \mapsto \sum_{|k| < 2\beta(R)} \hat{f}(k) e^{2\pi i kt}. \quad (5.14)$$

Proposition 11. Let f be a periodic signal and $\{\check{\chi}_R\}_{R=1}^\infty$ defined as in Proposition 10 and T_R as in Definition 2. Then

$$(S_{\check{\chi}_R} T_R f) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) = (-1)^\tau \sqrt{\beta(p)} f_{p,\tau}, \quad \tau = 0, \dots, \beta(p) - 1, \quad |p| \leq R, \quad (5.15)$$

where

$$f_{p,\tau} = (f, D_{p,\tau})_{L^2([0,1])}.$$

Proof. Since the functions $(D_{p,\tau})$ form an orthonormal basis of $L^2([0, 1])$, we have

$$f(t) = \sum_{p',\tau'} (f, D_{p',\tau'})_{L^2([0,1])} D_{p',\tau'}(t), \quad \text{a.e.}$$

Notice that

$$(T_R f)(t) = \sum_{|p'| \leq R} \sum_{\tau'=0}^{\beta(p')-1} f_{p',\tau'} D_{p',\tau'}(t),$$

where $f_{p',\tau'} = (f, D_{p',\tau'})_{L^2([0,1])}$. By linearity,

$$(S_{\check{\chi}_R} T_R f) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) = \sum_{|p'| \leq R} \sum_{\tau'=0}^{\beta(p')-1} f_{p',\tau'} (S_{\check{\chi}_R} D_{p',\tau'}) \left(\frac{\tau}{\beta(p)}, \nu(p) \right).$$

If $|p| \leq R$, by Proposition 10,

$$(S_{\check{\chi}_R} T_R f) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) = \sum_{\tau'=0}^{\beta(p)-1} f_{p,\tau'} (S_{\check{\chi}_R} D_{p,\tau'}) \left(\frac{\tau}{\beta(p)}, \nu(p) \right). \quad (5.16)$$

In Corollary 1 we proved that

$$D_{p,\tau'} \left(\frac{\tau}{\beta(p)} \right) = \sqrt{\beta(p)} \delta_0(\tau - \tau').$$

Therefore, by Proposition 10, (5.16) turns into

$$\begin{aligned} (S_{\check{\chi}_R} T_R f) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) &= e^{-2\pi i \nu(p) \frac{\tau}{\beta(p)}} f_{p,\tau} D_{p,\tau} \left(\frac{\tau}{\beta(p)} \right) \\ &= e^{-2\pi i \nu(p) \tau / \beta(p)} \sqrt{\beta(p)} f_{p,\tau}. \end{aligned}$$

Since $\nu(p) = \pm 3/2 \beta(p)$,

$$e^{-2\pi i \nu(p) \tau / \beta(p)} = e^{\mp 3\pi i \tau} = (-1)^\tau.$$

Therefore, finally,

$$(S_{\check{\chi}_R} T_R f) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) = (-1)^\tau \sqrt{\beta(p)} f_{p,\tau}, \quad |p| \leq R. \quad \square \tag{5.17}$$

The definition of $\check{\chi}_R$ in (5.11) implies that

$$\|\check{\chi}_R - \check{\chi}\|_Z \rightarrow 0.$$

Moreover, it is immediate that, for all $f \in L^2([0, 1])$, $\|T_R f - f\|_{L^2([0,1])} \rightarrow 0$. Therefore, by the continuity properties of S , proven in Theorem 8, for all $f \in L^2([0, 1])$

$$\|(S_{\check{\chi}_R} T_R f) - (S_{\check{\chi}} f)\|_Y \rightarrow 0, \quad R \rightarrow \infty. \tag{5.18}$$

Eqs. (5.17) and (5.18) clarify the representation of the S-transform of a periodic signal f via the Stockwell coefficients $f_{p,\tau}$. Moreover, (5.17) explains the role of the multiplicative factor $(-1)^\tau$ in front of the basis functions $D_{p,\tau}$ used by R.G. Stockwell in [27].

Remark 2. In the paper we have always considered a symmetric partition of the frequency from the positive and negative side. Actually, the algorithm is slightly different: see [31–33] for details.

6. Window adapted basis construction

In this section we determine a basis of $L^2([0, 1])$ adapted to an admissible window φ satisfying Assumption 1. As explained in the introduction, we want to find a basis E_p^φ such that $S_\varphi E_p^\varphi$ is *local* both in time and in frequency and such that the evaluation of all coefficients $f_p^\varphi = (f, E_p^\varphi)_{L^2([0,1])}$ is fast: $\mathcal{O}(N \log N)$. In Section 3, we proved that $D_{p,\tau}$ is a basis of $L^2([0, 1])$ which is *local* both in time and in frequency. Moreover, in Section 5, we showed that the natural discretization of the time–frequency domain in this setting is given by the dyadic decomposition in the frequency domain and the $\tau/\beta(p)$ grid in the time domain. So, it is natural to change our task in finding a basis $E_{p,\tau}^\varphi$ such that (see Fig. 3)

$$(S_\varphi E_{p,\tau}^\varphi)(b, \nu(p)) = e^{-2\pi i b \nu(p)} D_{p,\tau}(b). \tag{6.1}$$

As in the previous section, in order to obtain the punctual evaluation (6.1), we introduce a sequence $\{\varphi_R\}_{R=1}^\infty$ as in Lemma 3. In order to keep the notation easier, we set

$$c_{p,j}^\varphi(\nu(p)) = c_{p,j}^{\varphi_R}(\nu(p)), \quad |p| \leq R.$$

Notice that this definition makes sense in view of (5.4).

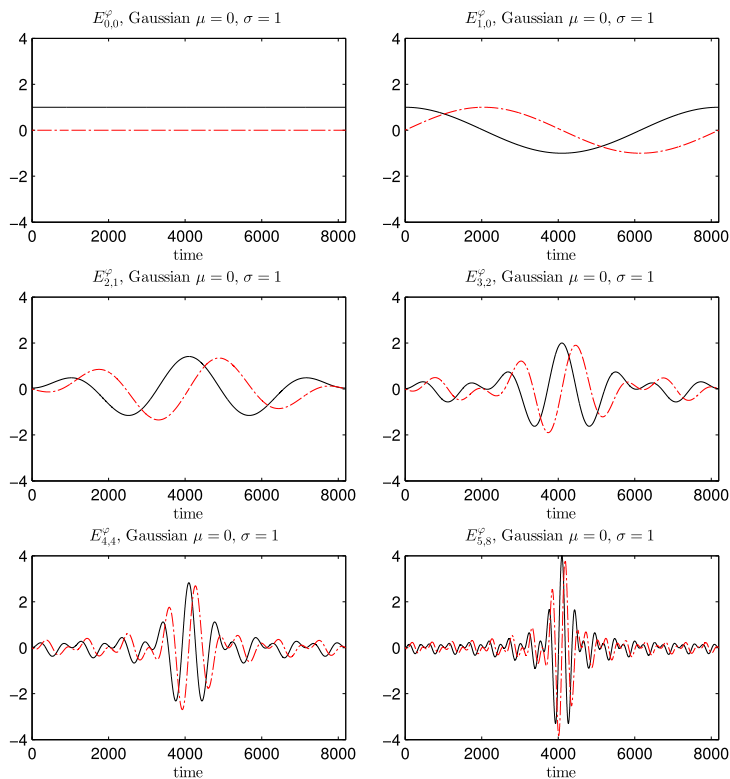


Fig. 3. $E_{p,\tau}^\varphi$ basis functions in increasing frequency p -bands. Black line = real, red line = imaginary. $\widehat{\varphi}$ is a truncated Gaussian window with $\mu = 0$ and $\sigma = 1$. Notice the similarities with Fig. 1. Indeed, in this case the ratio $(\delta/M)^2$ is approximately 0.8836 and $(M/\delta)^2$ is approximately 1.13173, see Theorem 14.

Theorem 12. Let φ be an admissible window satisfying Assumption 1 and

$$E_{p,\tau}^\varphi(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} [c_{p,j}^\varphi(\nu(p))]^{-1} e^{2\pi i (\beta(p)+j)(t - \frac{\tau}{\beta(p)})}. \tag{6.2}$$

Then

$$(S_{\varphi_R} E_{p,\tau}^\varphi)(b, \nu(p)) = e^{-2\pi i b \nu(p)} D_{p,\tau}(b), \quad |p| \leq R. \tag{6.3}$$

Moreover,

$$\bigcup_{p \in \mathbb{Z}} E_p^\varphi,$$

where

$$E_p^\varphi = \{E_{p,\tau}^\varphi\}_{\tau=0, \dots, \beta(|p|)-1}$$

is a basis of $L^2([0, 1])$.

Remark 3. Take $\check{\chi}$ as in (5.11), then

$$c_{p,j}^{\check{\chi}}(\nu(p)) = 1,$$

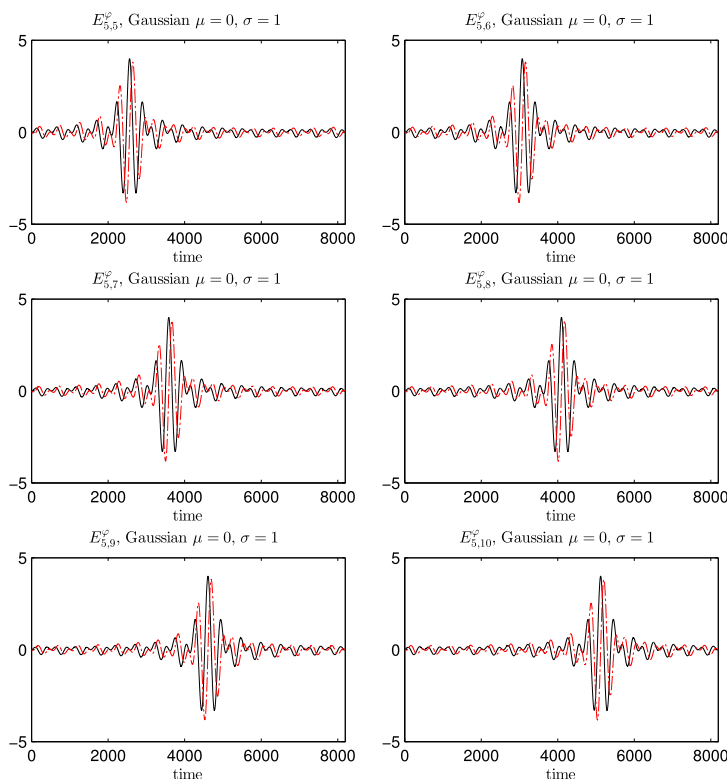


Fig. 4. $E_{p,\tau}^\varphi$ basis functions in the same frequency p -band ($p = 5$). Black line = real, red line = imaginary. $\widehat{\varphi}$ is a truncated Gaussian window with $\mu = 0$ and $\sigma = 1$. See Fig. 2 for comparison.

for all p and j . So, by (6.2) and (3.2),

$$\begin{aligned} E_{p,\tau}^{\check{X}}(t) &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \left[c_{p,j}^{\check{X}}(\nu(p)) \right]^{-1} e^{2\pi i (\beta(p)+j) \left(t - \frac{\tau}{\beta(p)} \right)} \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j) \left(t - \frac{\tau}{\beta(p)} \right)} \\ &= D_{p,\tau}(t). \end{aligned}$$

Hence, the functions $E_{p,\tau}^\varphi$ are a proper generalization of the DOST functions (see Figs. 4 and 5).

Proof. By Eq. (4.8), it follows that the functions $E_{p,\tau}^\varphi$ do satisfy (6.3). So, we only need to prove that $\bigcup_{p \in \mathbb{Z}} E_p^\varphi$ is a basis of $L^2([0, 1])$.

Notice that

$$E_p^\varphi \subseteq \text{span} \left\{ e^{2\pi i kt} \right\}_{k \in [\beta(p), 2\beta(p)-1]} = \text{span} \{ D_{p,\tau} \}_{\tau=0, \dots, \beta(p)-1}.$$

It is sufficient to check that E_p^φ is a linear independent set. Let us assume that there exist $\{\alpha_\tau\}_{\tau=0}^{\beta(p)-1}$ such that

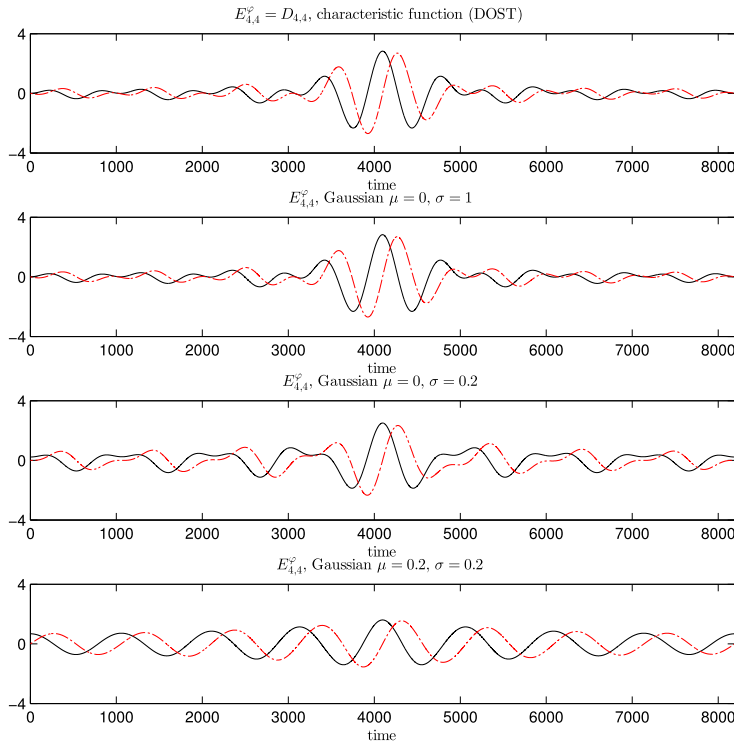


Fig. 5. $E_{p,\tau}^\varphi$ basis functions with $p = 4$ and $\tau = 4$ with different windows. Black line = real, red line = imaginary. The Fourier transform of $\widehat{\varphi}$ is $\chi_{(-1/3,1/3)}$ in the first plot, then a truncated Gaussian with varying μ and σ .

$$\sum_{\tau=0}^{\beta(p)-1} \alpha_\tau E_{p,\tau}^\varphi(t) = 0.$$

Then, by (6.3), for $|p| \leq R$, we obtain

$$\begin{aligned} 0 &= \left(S_{\varphi_R} \sum_{\tau=0}^{\beta(p)-1} \alpha_\tau E_{p,\tau}^\varphi \right) (b, \nu(p)) \\ &= \sum_{\tau=0}^{\beta(p)-1} \alpha_\tau (S_{\varphi_R} E_{p,\tau}^\varphi) (b, \nu(p)) \\ &= e^{-2\pi i b \nu(p)} \sum_{\tau=0}^{\beta(p)-1} \alpha_\tau D_{p,\tau}(b). \end{aligned}$$

Hence,

$$\sum_{\tau=0}^{\beta(p)-1} \alpha_\tau D_{p,\tau}(b) = 0. \tag{6.4}$$

Since $D_{p,\tau}$ is a basis, (6.4) implies that α_τ are all zeros. That is, $E_{p,\tau}^\varphi$ are linear independent. \square

See Figs. 3, 4 for an example of an $E_{p,\tau}^\varphi$ basis with $\widehat{\varphi}$ equal to a truncated gaussian. In Fig. 5 there is the comparison of function $E_{4,4}^\varphi$ with different window φ . In Fig. 6 is represented the decomposition of a given test signal with respect with different windowed basis $E_{p,\tau}^\varphi$.

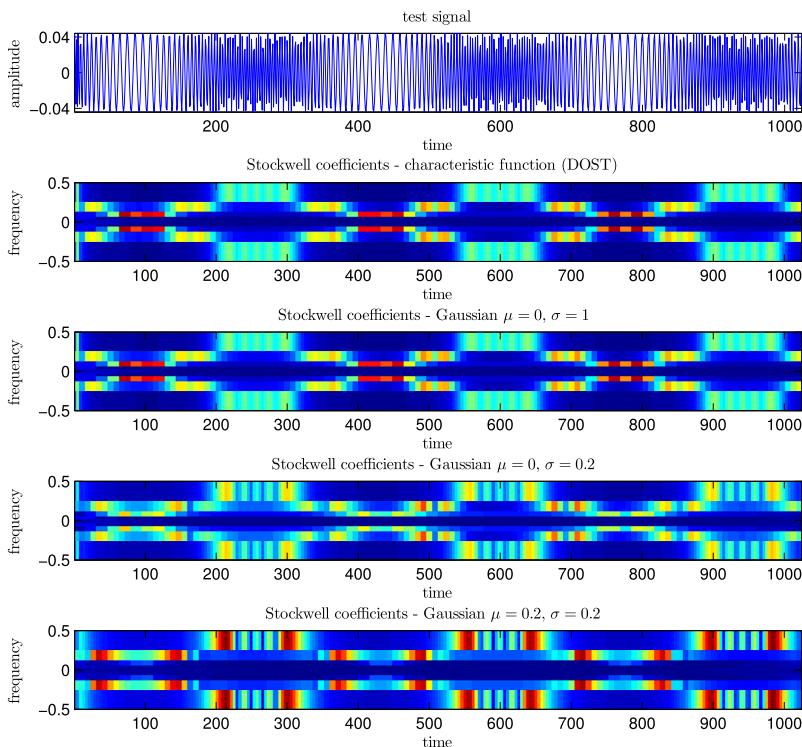


Fig. 6. Decompositions of a given test signal on different windowed basis.

Proposition 13. Let $E_{p,\tau}^\varphi$ as in Theorem 12 and let f be a finite signal. Then the evaluation of the coefficients

$$f_{p,\tau}^\varphi = (f, E_{p,\tau}^\varphi)_{L^2([0,1])}$$

has computational complexity $\mathcal{O}(N \log N)$, where N is the length of f .

Proof. By Plancharel’s Theorem we can write

$$f_{p,\tau}^\varphi = (f, E_{p,\tau}^\varphi)_{L^2([0,1])} = \left(\hat{f}, \widehat{E_{p,\tau}^\varphi} \right)_{l^2(\mathbb{Z})}.$$

Using the explicit expression of the basis $E_{p,\tau}^\varphi$, we obtain

$$\begin{aligned} f_{p,\tau}^\varphi &= \left(\hat{f}, \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} [c_{p,j}^\varphi(\nu(p))]^{-1} e^{-2\pi i (\beta(p)+j)(\tau/\beta(p))} \delta_{\beta(p)+j}(\cdot) \right)_{l^2(\mathbb{Z})} \\ &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \hat{f}(\beta(p)+j) \overline{[c_{p,j}^\varphi(\nu(p))]^{-1}} e^{2\pi i (\beta(p)+j)(\tau/\beta(p))} \\ &= \left(R^\varphi \hat{f}, \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{-2\pi i (\beta(p)+j)(\tau/\beta(p))} \delta_{\beta(p)+j}(\cdot) \right)_{l^2(\mathbb{Z})} \\ &= \left(F^{-1} R^\varphi \hat{f}, D_{p,\tau} \right)_{L^2([0,1])}, \end{aligned}$$

where R^φ is a sequence in \mathbb{Z} such that

$$R^\varphi(\beta(p) + j) = \overline{[c_{p,j}^\varphi(\nu(p))]}^{-1} \tag{6.5}$$

for all p and related j . Hence,

$$f_{p,\tau}^\varphi = (f, E_{p,\tau}^\varphi)_{L^2([0,1])} = (\tilde{f}, D_{p,\tau})_{L^2([0,1])} \tag{6.6}$$

where $\tilde{f} = F^{-1} R^\varphi \hat{f}$. Given \tilde{f} , computing (6.6) using the FDOT algorithm introduced in [33] has complexity $\mathcal{O}(N \log N)$ and computing \tilde{f} via FFT has complexity $\mathcal{O}(N \log N)$. So, the computational complexity remains $\mathcal{O}(N \log N)$ (see Fig. 6). \square

Remark 4. It is worth checking explicitly the computational complexity of the algorithm. To perform this task, we start evaluating the column vector f_p^φ given by

$$\begin{aligned} f_p^\varphi &= \{f_{p,\tau}^\varphi\}_{\tau=0}^{\beta(p)-1} \\ &= \left\{ (f, E_{p,\tau}^\varphi)_{L^2([0,1])} \right\}_{\tau=0}^{\beta(p)-1} \\ &= \left\{ (\hat{f}, \widehat{E_{p,\tau}^\varphi})_{l^2(\mathbb{Z})} \right\}_{\tau=0}^{\beta(p)-1} \\ &= \left\{ \left(\hat{f}, \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} [c_{p,j}^\varphi(\nu(p))]^{-1} e^{-2\pi i (\beta(p)+j)(\tau/\beta(p))} \delta_{\beta(p)+j}(\cdot) \right)_{l^2(\mathbb{Z})} \right\}_{\tau=0}^{\beta(p)-1} \\ &= \left\{ \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \hat{f}(\beta(p) + j) \overline{[c_{p,j}^\varphi(\nu(p))]}^{-1} e^{2\pi i (\beta(p)+j)(\tau/\beta(p))} \right\}_{\tau=0}^{\beta(p)-1} \\ &= \left\{ \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \hat{f}(\beta(p) + j) \overline{[c_{p,j}^\varphi(\nu(p))]}^{-1} e^{2\pi i j(\tau/\beta(p))} \right\}_{\tau=0}^{\beta(p)-1} \\ &= (F_{j \rightarrow \tau}^{-1} \left((R^\varphi \hat{f})|_{\beta(p), \dots, 2\beta(p)-1}(j) \right))(\tau) \end{aligned}$$

where R^φ is defined as in (6.5). Therefore, first we have to perform the FFT of the signal f ($\mathcal{O}(N \log N)$), and the multiplication by R^φ ($\mathcal{O}(N)$), then at each p band we need to use the FFT to perform the anti Fourier transform with computational complexity $\mathcal{O}(\beta(p) \log \beta(p))$. Summing up the contribution of each p -band we obtain the computational complexity of $\mathcal{O}(N \log N)$.

The basis $\{E_{p,\tau}^\varphi\}_{p,\tau}$ is in general not orthogonal nor normal. Nevertheless, we can normalize it setting

$$F_{p,\tau}^\varphi(t) = \frac{E_{p,\tau}^\varphi(t)}{\|E_{p,\tau}^\varphi\|_{L^2([0,1])}}, \tag{6.7}$$

so that

$$\|F_{p,\tau}^\varphi\|_{L^2([0,1])} = 1.$$

Notice that

$$\|E_{p,\tau}^\varphi\|_{L^2([0,1])} = \|E_{p,\tau'}^\varphi\|_{L^2([0,1])} = N_p^\varphi \tag{6.8}$$

depends just on the p -band, not on τ .

The basis $\{F_{p,\tau}^\varphi(t)\}_{p,\tau}$ fails in general to be orthogonal. Nevertheless, assuming a mild condition on φ , we can prove that it is a frame.

Theorem 14. *Let φ be an admissible window function satisfying Assumption 1, and such that*

$$\inf_{\xi \in (-1/3, 1/3)} |\overline{\widehat{\varphi}(\xi)}| \geq \delta > 0 \tag{6.9}$$

$$\sup_{\xi \in (-1/3, 1/3)} |\overline{\widehat{\varphi}(\xi)}| \leq M < \infty \tag{6.10}$$

then the basis $\bigcup_{p \in \mathbb{Z}} F_p^\varphi$ is a frame of $L^2([0, 1])$, where

$$F_p^\varphi = \{F_{p,\tau}^\varphi\}_{\tau=0, \dots, \beta(|p|)}.$$

In particular

$$\left(\frac{\delta}{M}\right)^2 \|f\|_{L^2([0,1])}^2 \leq \sum_{p,\tau} |(f, F_{p,\tau}^\varphi)_{L^2([0,1])}|^2 \leq \left(\frac{M}{\delta}\right)^2 \|f\|_{L^2([0,1])}^2.$$

Proof. Notice that under the hypothesis (6.9), (6.10), by (6.8)

$$\frac{1}{M} \leq N_p^\varphi \leq \frac{1}{\delta}, \quad \forall p \in \mathbb{Z}. \tag{6.11}$$

Observe, by a slight variation of (6.6), that

$$(f, F_{p,\tau}^\varphi)_{L^2([0,1])} = (F^{-1} \widetilde{R}^\varphi \hat{f}, D_{p,\tau})_{L^2([0,1])}$$

where \widetilde{R}^φ is a sequence such that

$$\widetilde{R}^\varphi(\beta(p) + j) = \frac{R^\varphi(\beta(p) + j)}{N_p^\varphi} = \frac{[c_{p,j}^\varphi(\nu(p))]^{-1}}{N_p^\varphi},$$

where N_p^φ is as in (6.8).

If the window φ satisfies condition (6.10), by (6.11), we have

$$\sup_{k \in \mathbb{Z}} \left\{ |\widetilde{R}^\varphi(k)| \right\} \leq \frac{M}{\delta} < \infty, \tag{6.12}$$

$$\inf_{k \in \mathbb{Z}} \left\{ |\widetilde{R}^\varphi(k)| \right\} \geq \frac{\delta}{M} > 0. \tag{6.13}$$

Hence, since $\bigcup_{p \in \mathbb{Z}} D_p$ is an orthonormal basis and since F is a unitary operator from $L^2([0, 1])$ to $l^2(\mathbb{Z})$, we obtain

$$\begin{aligned} \sum_{p,\tau} |(f, F_{p,\tau}^\varphi)_{L^2([0,1])}|^2 &= \sum_{p,\tau} \left| (F^{-1} \widetilde{R}^\varphi \hat{f}, D_{p,\tau})_{L^2([0,1])} \right|^2 = \|F^{-1} \widetilde{R}^\varphi \hat{f}\|_{L^2([0,1])}^2 \\ &= \|\widetilde{R}^\varphi \hat{f}\|_{l^2(\mathbb{Z})}^2 \leq \left(\sup_{k \in \mathbb{Z}} \left\{ |\widetilde{R}^\varphi(k)| \right\} \right)^2 \|\hat{f}\|_{l^2(\mathbb{Z})}^2 \leq \left(\frac{M}{\delta}\right)^2 \|f\|_{L^2([0,1])}^2, \end{aligned}$$

and

$$\begin{aligned} \sum_{p,\tau} \left| (f, F_{p,\tau}^\varphi)_{L^2([0,1])} \right|^2 &= \sum_{p,\tau} \left| (F^{-1} \widetilde{R}^\varphi \hat{f}, D_{p,\tau})_{L^2([0,1])} \right|^2 = \left\| F^{-1} \widetilde{R}^\varphi \hat{f} \right\|_{L^2([0,1])}^2 \\ &= \left\| \widetilde{R}^\varphi \hat{f} \right\|_{l^2(\mathbb{Z})} \geq \left(\inf_{k \in \mathbb{Z}} \left\{ \left| \widetilde{R}^\varphi(k) \right| \right\} \right)^2 \left\| \hat{f} \right\|_{l^2(\mathbb{Z})}^2 \geq \left(\frac{\delta}{M} \right)^2 \|f\|_{L^2([0,1])}^2. \quad \square \end{aligned}$$

Since $\bigcup_{p \in \mathbb{Z}} F_p^\varphi$ forms a frame, it is possible to obtain abstractly the canonical dual frame, we denote $\widetilde{F}_{p,\tau}^\varphi$. So, following the same scheme of Proposition 11 and Eq. (6.3), we have

$$\begin{aligned} (S_{\varphi_R} T_R f) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) &= \left(S_{\varphi} T_R \sum_{p',\tau'} \left(f, \widetilde{F}_{p',\tau'}^\varphi \right)_{L^2([0,1])} F_{p',\tau'}^\varphi \right) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) \\ &= \sum_{|p'| \leq R} \sum_{\tau'=0}^{\beta(p')-1} \left(f, \widetilde{F}_{p',\tau'}^\varphi \right)_{L^2([0,1])} \left(S_{\varphi_R} F_{p',\tau'}^\varphi \right) \left(\frac{\tau}{\beta(p)}, \nu(p) \right) \\ &= \sum_{p' \leq R} \sum_{\tau'=0}^{\beta(p')-1} \left(f, \widetilde{F}_{p',\tau'}^\varphi \right)_{L^2([0,1])} \frac{e^{-2\pi i \frac{\tau}{\beta(p)} \nu(p')}}{N_{p'}} D_{p',\tau'} \left(\frac{\tau}{\beta(p)}, \nu(p) \right) \\ &= (-1)^\tau \sqrt{\beta(p)} \frac{\left(f, \widetilde{F}_{p,\tau}^\varphi \right)_{L^2([0,1])}}{N_p}, \quad |p| \leq R. \end{aligned}$$

Remark 5. Notice that, when in Eqs. (6.9) and (6.10) $\delta = M$, we have a tight-frame. In the case of the DOST basis, i.e. $E_{p,\tau}^\chi = D_{p,\tau}$ it is clear that $\delta = M = 1$. So $D_{p,\tau}$ is a tight-frame. Actually, we have proven more: $D_{p,\tau}$ is an orthonormal basis.

Acknowledgments

The authors thank M. Berra, P. Boggiatto, E. Cordero, V. Giannini, M. Lupini, F. Nicola, A. Vignati, M.W. Wong and H. Zhu for fruitful discussions and comments. We are grateful to the referees for a number of helpful suggestions for improvement in the article, in particular for remarks on Theorem 8 and Theorem 14.

The first author has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), grant title *Analisi Tempo-Frequenza ed Applicazioni*, and by a Postdoc scholarship of the Università degli Studi di Torino.

The second author was supported by grants from the Doctoral School of Sciences and Innovative Technologies, Ph.D. Program in Mathematics of Università degli Studi di Torino.

References

- [1] P. Balazs, M. Dörfler, F. Jaillet, N. Holighaus, G. Velasco, Theory, implementation and applications of nonstationary Gabor frames, *J. Comput. Appl. Math.* 236 (6) (2011) 1481–1496.
- [2] D. Bayer, M. Speckbacher, Coorbit theory of α -modulation spaces, in: *Communication at the Conference Modern Time-Frequency Analysis*, Strobl, 2014.
- [3] J.J. Benedetto, S. Li, The theory of multiresolution analysis frames and applications to filter banks, *Appl. Comput. Harmon. Anal.* 5 (4) (1998) 389–427.
- [4] M. Biswal, P.K. Dash, Detection and characterization of multiple power quality disturbances with a fast S-transform and decision tree based classifier, *Digit. Signal Process.* 23 (4) (2013) 1071–1083.

- [5] P. Boggiatto, E. Carypis, A. Oliaro, Two aspects of the Donoho–Stark uncertainty principle, to appear.
- [6] P. Boggiatto, C. Fernandez, A. Galbis, A group representation related to the Stockwell transform, *Indiana Univ. Math. J.* 58 (5) (2009) 2277–2304.
- [7] R.S. Choraś, Time–frequency analysis of image based on Stockwell transform, in: *Image Processing and Communications Challenges 5*, in: *Advances in Intelligent Systems and Computing*, vol. 233, Springer International Publishing, 2014, pp. 91–97.
- [8] S. Dahlke, M. Fornasier, H. Rauhut, G. Steidl, G. Teschke, Generalized coorbit theory, Banach frames, and the relation to α -modulation spaces, *Proc. Lond. Math. Soc.* (3) 96 (2) (2008) 464–506.
- [9] I. Daubechies, et al., *Ten Lectures on Wavelets*, vol. 61, SIAM, 1992.
- [10] I. Djurović, E. Sejdić, J. Jiang, Frequency-based window width optimization for S-transform, *AEÜ, Int. J. Electron. Commun.* 62 (4) (2008) 245–250.
- [11] D.L. Donoho, P.B. Stark, Uncertainty principles and signal recovery, *SIAM J. Appl. Math.* 49 (3) (1989) 906–931.
- [12] S. Drabycz, R.G. Stockwell, J.R. Mitchell, Image texture characterization using the discrete orthonormal S-transform, *J. Digit. Imaging* 22 (6) (2009) 696–708.
- [13] H.G. Feichtinger, K. Gröchenig, Gabor frames and time–frequency analysis of distributions, *J. Funct. Anal.* 146 (2) (1997) 464–495.
- [14] M. Fornasier, Banach frames for α -modulation spaces, *Appl. Comput. Harmon. Anal.* 22 (2) (2007) 157–175.
- [15] P.C. Gibson, M.P. Lamoureux, G.F. Margrave, Letter to the editor: Stockwell and wavelet transforms, *J. Fourier Anal. Appl.* 12 (6) (2006) 713–721.
- [16] B.G. Goodyear, H. Zhu, R.A. Brown, J.R. Mitchell, Removal of phase artifacts from fMRI using a Stockwell transform filter improves brain activity detection, *Magn. Reson. Med.* 51 (2004) 16–21.
- [17] K. Gröchenig, *Foundations of Time–Frequency Analysis*, *Appl. Numer. Harmon. Anal.*, Birkhäuser Boston, Inc., Boston, MA, 2001.
- [18] Q. Guo, S. Molahajloo, M.W. Wong, Modified Stockwell transforms and time–frequency analysis, in: *New Developments in Pseudo-Differential Operators*, in: *Oper. Theory Adv. Appl.*, vol. 189, Birkhäuser, Basel, 2009, pp. 275–285.
- [19] C. Han-Peng, H. Zhen-Hua, H. De-Ji, Seismic data denoising based on mixed time–frequency methods, *J. Appl. Geophys.* 8 (4) (2011) 319–327.
- [20] M. Jaya Bharata Reddy, R. Krishnan Raghupathy, K.P. Venkatesh, D.K. Mohanta, Power quality analysis using Discrete Orthogonal S-transform (DOST), *Digit. Signal Process.* 23 (2) (2013) 616–626.
- [21] C. Kalisa, B. Torrèsani, N -dimensional affine Weyl–Heisenberg wavelets, *Ann. Inst. Henri Poincaré, a Phys. Théor.* 59 (2) (1993) 201–236.
- [22] J. Ladan, *An analysis of Stockwell transforms, with applications to image processing*, Thesis, University of Waterloo, Ontario, Canada, 2014.
- [23] S. Mallat, *A Wavelet Tour of Signal Processing*, 3rd edn., Elsevier/Academic Press, Amsterdam, 2009.
- [24] N. Ortigosa, O. Cano, G. Ayala, A. Galbis, C. Fernández, Atrial fibrillation subtypes classification using the general Fourier-family transform, *Med. Eng. Phys.* 36 (4) (2014) 554–560.
- [25] L. Riba, *Multi-dimensional Stockwell transforms and applications*, PhD thesis, Università degli Studi di Torino, Italy, 2014.
- [26] L. Riba, M. Wong, Continuous inversion formulas for multi-dimensional modified Stockwell transforms, *Integral Transforms Spec. Funct.* 26 (1) (2015) 9–19.
- [27] R.G. Stockwell, A basis for efficient representation of the S-transform, *Digit. Signal Process.* 17 (2007) 371–393.
- [28] R.G. Stockwell, Why use the S-transform?, in: *Pseudo-Differential Operators: Partial Differential Equations and Time–Frequency Analysis*, in: *Fields Inst. Commun.*, vol. 52, Amer. Math. Soc., Providence, RI, 2007, pp. 279–309.
- [29] R.G. Stockwell, L. Mansinha, R.P. Lowe, Localization of the complex spectrum: the S transform, *IEEE Trans. Signal Process.* 44 (1996) 998–1001.
- [30] S. Ventosa, C. Simon, M. Schimmel, J.J. Dañobeitia, A. M̀manuel, The S-transform from a wavelet point of view, *IEEE Trans. Signal Process.* 56 (7, part 1) (2008) 2771–2780.
- [31] Y. Wang, *Efficient Stockwell transform with applications to image processing*, PhD thesis, University of Waterloo, Ontario, Canada, 2011.
- [32] Y. Wang, J. Orchard, Symmetric Discrete Orthonormal Stockwell Transform, *American Institute of Physics (AIP) Conference Proceedings*, vol. 1048, 2008, pp. 585–588.
- [33] Y. Wang, J. Orchard, Fast discrete orthonormal Stockwell transform, *SIAM J. Sci. Comput.* 31 (2009) 4000–4012.
- [34] M.W. Wong, H. Zhu, A characterization of Stockwell spectra, in: *Modern Trends in Pseudo-Differential Operators*, in: *Oper. Theory Adv. Appl.*, vol. 172, Birkhäuser, Basel, 2007, pp. 251–257.
- [35] Y. Yan, H. Zhu, The generalization of discrete Stockwell transform, in: *19th European Signal Processing Conference, EUSIPCO 2011*, 2011.
- [36] H. Zhu, B. Goodyear, M. Lauzon, R. Brown, G. Mayer, A. Law, L. Mansinha, J. Mitchell, A new local multiscale Fourier analysis for medical imaging, *Med. Phys.* 30 (6) (2003) 1134–1141.